

Dense nuclear Fréchet ideals in C^* -algebras

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Abstract

We show that a C^* -algebra B contains a dense left or right Fréchet ideal A , with A a nuclear locally convex space, if and only if the primitive ideal space $\text{Prim}(B)$ of B is discrete and countable, and B/I is finite dimensional for each $I \in \text{Prim}(B)$. Here $\{\|\cdot\|_n\}_{n=0}^\infty$ denotes a family of increasing norms topologizing A . We show the forward implication holds for a general Banach algebra B , if the ideal is assumed two-sided. For C^* -algebras, we construct dense nuclear ideals by defining a set of matrix-valued Schwartz functions on the countable discrete space $\text{Prim}(B)$. AMS Subject Classification 2010: 46H20 (structure, classification of topological algebras), 46H10 (ideals and subalgebras). Keywords: Nuclear Fréchet space, C^* -algebra, discrete spectrum, dense ideal.

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1 Introduction

Dense subalgebras of C^* -algebras are well-known to be useful in the study of C^* -algebras. The dense subalgebra can be viewed as C^∞ functions on a manifold, where instead of a manifold we have an underlying “noncommutative space”. To be more useful, the dense subalgebra often has a Fréchet topology and is *spectral invariant* in the C^* -algebra (see Remark 3.2). While some C^* -algebras have nice dense subalgebras, others don’t seem to.

If we insist that the dense subalgebra be an *ideal* in the C^* -algebra, we get a much stronger condition than spectral invariance. Few C^* -algebras can have a “smooth” dense ideal. For example, any compact manifold M without boundary has a spectral invariant Fréchet algebra of smooth functions $C^\infty(M)$, dense in the C^* -algebra of continuous functions $C(M)$. But $C^\infty(M)$ is an ideal in $C(M)$, if and only if M is discrete (and therefore finite by compactness).

In this paper, I will classify which C^* -algebras have dense nuclear Fréchet ideals. The nuclearity property plays the role of making elements of the subalgebra “smooth” or “differentiable”. For example, $C^\infty(M)$ is a nuclear Fréchet space [Treves, 1967], Chapters 10 and 51.

The notion of nuclearity for a locally convex space is different than nuclearity for a C^* -algebra [Kad Ring II, 1997], Chapter 11. If a C^* -algebra were nuclear as a locally convex space, it would be finite dimensional (Proposition 2.1 (b) below). However, it is reasonable that a dense Fréchet subalgebra of an infinite dimensional nuclear C^* -algebra be a nuclear locally convex space (see Corollary 4.7 below).

In §2, we recall definitions and properties of nuclear locally convex spaces and Fréchet

spaces from the literature. We define Schwartz functions on a countable set. Every nuclear Fréchet space with basis is one of these.

In §3, we go over definitions and basic lemmas on dense Fréchet ideals of Banach algebras, and state the example of $l^p(\mathbb{N})$ and Schwartz functions $\mathcal{S}(\mathbb{N})$ as dense ideals of the commutative pointwise-multiplication C^* -algebra $c_0(\mathbb{N})$. We show the property of having a dense nuclear ideal is preserved by taking quotients and subalgebras formed using idempotents.

In §4, we apply nuclearity and results of §3 to see that a C^* -algebra has a dense nuclear left or right ideal only if its spectrum is discrete and countable. We do this by showing that the “finite socle” is dense. In the commutative case, a shorter proof is given.

In §5, we generalize our results on discrete spectrum to the case of an arbitrary Banach algebra. We prove that any Banach algebra with a dense nuclear two-sided ideal is both left and right completely continuous. We also show the primitive ideal space is countable, and that primitive quotients are finite dimensional. Complete continuity on both sides of a Banach algebra is already known to imply discrete spectrum and finite dimensional primitive quotients [Kaplansky, 1949], [Kaplansky, 1948], Lemma 4.

In §6, we construct dense nuclear two-sided ideals for every C^* -algebra B which is the direct sum of simple finite dimensional C^* -algebras (i.e. full matrix algebras). We define our dense ideal to be matrix-valued Schwartz functions on the countable discrete spectrum of B . In many cases, the underlying Fréchet space structure of the dense ideal

is the standard Schwartz functions on \mathbb{N} , namely

$$\mathcal{S}(\mathbb{N}) = \{\varphi: \mathbb{N} \rightarrow \mathbb{C} \mid (1+k)^n |\varphi(k)| < \infty, k, n = 0, 1, 2, \dots\}. \quad (1)$$

In §7, we give examples of dense nuclear ideals, including the convolution algebra of C^∞ functions on a compact Lie group, and C^∞ functions on the Cantor set.

In Appendix A, we show that the constants m_n and C_n (in the dense ideal inequality (12)) can be made to satisfy $m_n = n$ and $C_n = 1$ by choosing an equivalent family of increasing norms $\{\|\cdot\|_n\}_{n=0}^\infty$ for the topology of the dense Fréchet ideal.

In Appendix B, we give counterexamples to our theorems, when various hypotheses are dropped.

All algebras in this paper are over the field of complex numbers \mathbb{C} . The set of natural numbers $\{0, 1, 2, \dots\}$ is denoted by \mathbb{N} , and $\mathbb{N}^+ = \{1, 2, 3, \dots\}$.

2 Nuclearity and Fréchet Spaces with Basis

We state some basic facts about nuclear Fréchet spaces used throughout the paper. The reader unfamiliar with these concepts can consult standard references, such as [Pietsch, 1972], [Treves, 1967].

Proposition 2.1. Nuclearity Facts.

- (a) *A nuclear Fréchet space is separable.*
- (b) *A nuclear Banach space must be finite dimensional.*

(c) *If a Fréchet space F contains an infinite dimensional Banach space E , and the topology on E inherited from F is stronger than the Banach space topology, then F is not nuclear.*

(d) *A bounded set in a nuclear Fréchet space is relatively compact.*

Proof: For separability, see [Pietsch, 1972], Theorem 4.4.10. For finite dimensionality of a nuclear normed space, see [Pietsch, 1972], Theorem 4.4.14, or [Treves, 1967], Chapter 50-12, Corollary 2.

Let F be a Fréchet space containing an infinite dimensional Banach space E . Since the Banach space topology on E is assumed weaker than the Fréchet space topology inherited from F , these two topologies agree on E [Treves, 1967], Chapter 17-7, Corollary 2. But a subspace of a nuclear Fréchet space is nuclear [Pietsch, 1972], Proposition 5.1.5, or [Treves, 1967], Proposition 50.1 (50.3). E is not nuclear by (b), so therefore F cannot be nuclear.

A bounded set in a nuclear locally convex space is precompact by [Pietsch, 1972], 4.4.7, or [Treves, 1967], Proposition 50.2 (50.12). \square

Remark 2.2. The hypotheses of Proposition 2.1 (c) imply that E is a closed subspace of F , namely $\overline{E}^F = E$. Here is a counterexample when the Banach space is not a closed subspace of the Fréchet space. Let $F = l_u(\mathbb{N})$ be the nuclear Fréchet space of complex-valued sequences on \mathbb{N} (allowed to be unbounded), topologized by the seminorms $\|\varphi\|_n = \max_{k \leq n} |\varphi(k)|$ (see [Pietsch, 1972], 4.3.4 or [Treves, 1967], Theorem 51.1), and let E be the Banach space $l^p(\mathbb{N})$ for any $1 \leq p \leq \infty$.

We will often cite the following obvious Corollary to Proposition 2.1 (a).

Corollary 2.3. *A Banach space with a dense nuclear Fréchet subspace is separable.*

For any locally convex space E , recall that the *polar* M^0 of a set $M \subseteq E$ is defined to be the set

$$\{\varphi \in E' \mid |\varphi(m)| \leq 1, \quad m \in M\}. \quad (2)$$

Here E' denotes the topological dual of E , the set of linear functionals on E which are continuous. We will use the following consequence of the Uniform Boundedness Theorem in §5.

Theorem 2.4. Uniform Boundedness. *Let E be a Banach space with unit ball U . If $\{\varphi_n\}_{n=0}^\infty \subseteq E'$ is a sequence of continuous linear functionals converging to zero pointwise, then $\sup \{ \|\varphi_n\|_{U^0} \mid n \in \mathbb{N} \} < \infty$.*

Proof: See [Rudin, 1973], Theorem 2.6. □

Definition 2.5. Fréchet Space with Basis. A sequence $\{e_k\}_{k \in \mathbb{N}}$ in a Fréchet space F is a *basis* if every $f \in F$ has a unique series expansion $f = \sum_{k \in \mathbb{N}} f_k e_k$ which converges in F , where each f_k is a complex number. By [Husain, 1991], Chapter I, Theorem 4.3, any basis in a Fréchet space is a *Schauder basis*, meaning that the coordinate functional $f \mapsto f_k$ is a continuous linear map from F to \mathbb{C} for each $k \in \mathbb{N}$. If for every $n \in \mathbb{N}$, there is some $C > 0$ and $q \in \mathbb{N}$ such that $|f_k| \|e_k\|_n = \|f_k e_k\|_n \leq C \|f\|_q$, for all $f \in F$ and $k \in \mathbb{N}$, then the basis is *equicontinuous* [Pietsch, 1972], 10.1.2. According to [Pietsch, 1972], Theorem 10.1.2, every Schauder basis in a Fréchet space is equicontinuous. If F is

nuclear, an equicontinuous basis is *absolute* [Pietsch, 1972], 10.2.1. Then for each $n \in \mathbb{N}$ there exists $C > 0$ and $q \in \mathbb{N}$ for which $\sum_{k=0}^{\infty} \|f_k\| \|e_k\|_n \leq C \|f\|_q$, for all $f \in F$. Then the series $\sum_{k=0}^{\infty} f_k e_k$ converges absolutely, and in any ordering. For each e_k , let $e'_k: F \rightarrow \mathbb{C}$ denote the coordinate functional $e'_k(f) = f_k$, $f \in F$.

Definition 2.6. Basic Schwartz Spaces. If X is any set, we say that a function $\sigma: X \rightarrow [1, \infty)$ is a *scale* on X . If $\sigma = \{\sigma_n\}_{n=0}^{\infty}$ is a family of scales on X , define the Fréchet space

$$\mathcal{S}_{\sigma}^1(X) = \left\{ \varphi: X \rightarrow \mathbb{C} \mid \|\varphi\|_n^1 < \infty, n \in \mathbb{N} \right\}, \quad (3)$$

where

$$\|\varphi\|_n^1 = \sum_{\alpha \in X} \sigma_n(\alpha) |\varphi(\alpha)|. \quad (4)$$

Similarly, define $\mathcal{S}_{\sigma}^{\infty}(X)$ using sup-norms in place of ℓ^1 norms:

$$\|\varphi\|_n^{\infty} = \sup_{\alpha \in X} \left\{ \sigma_n(\alpha) |\varphi(\alpha)| \right\}. \quad (5)$$

We call $\mathcal{S}_{\sigma}^1(X)$ the ℓ^1 -norm σ -rapidly vanishing functions on X , and $\mathcal{S}_{\sigma}^{\infty}(X)$ the sup-norm σ -rapidly vanishing functions on X . Usually the family of scales σ will satisfy $\sigma_0 \leq \sigma_1 \leq \dots \sigma_n \leq \dots$, so that the families of norms $\{\|\cdot\|_n^1\}_{n=0}^{\infty}$ and $\{\|\cdot\|_n^{\infty}\}_{n=0}^{\infty}$ are increasing. When the algebra containing the Schwartz space is a C^* -algebra, we usually have $\sigma_0 \equiv 1$. Let δ_{α} denote the step function at $\alpha \in X$. Then $\text{span}\{\delta_{\alpha} \mid \alpha \in X\} = c_f(X)$ is the dense subspace of finite support functions in $\mathcal{S}_{\sigma}^1(X)$ and $\mathcal{S}_{\sigma}^{\infty}(X)$.¹ If X is a countable set, then

¹We define $\mathcal{S}_{\sigma}^{\infty}(X)$ as the completion of $c_f(X)$ in the sup norms (5), or, alternatively, restrict to functions vanishing at infinity when defining $\mathcal{S}_{\sigma}^{\infty}(X)$. This happens automatically when σ_n is *proper* for sufficiently large n . In either case $c_f(X)$ is dense.

$\{\delta_\alpha\}_{\alpha \in X}$ is an equicontinuous Schauder basis for $\mathcal{S}_\sigma^\infty(X)$ since $|\varphi(\alpha)|\|\delta_\alpha\|_n^\infty \leq \|\varphi\|_n^\infty$, and an equicontinuous absolute basis for $\mathcal{S}_\sigma^1(X)$ since $\sum_{\alpha \in X} |\varphi(\alpha)|\|\delta_\alpha\|_n^1 = \|\varphi\|_n^1$.

We say that a family of scales σ on the set X *dominates* another family β if for every $n \in \mathbb{N}$ there exists $C_n > 0$ and $m \in \mathbb{N}$ for which $\beta_n(\alpha) \leq C_n \sigma_m(\alpha)$ holds for all $\alpha \in X$, and write $\beta \lesssim \sigma$. If β also dominates σ , we say that σ and β are *equivalent*, and write $\sigma \sim \beta$. If σ is a single scale on X , a family of scales is given by $\sigma_n = \sigma^n$. One can easily verify that the identity map $\text{id}: c_f(X) \rightarrow c_f(X)$ extends to an isomorphism of Fréchet spaces $\mathcal{S}_\sigma^1(X) \cong \mathcal{S}_\beta^1(X)$ or $\mathcal{S}_\sigma^\infty(X) \cong \mathcal{S}_\beta^\infty(X)$ if and only if $\sigma \sim \beta$.

Theorem 2.7. Nuclear Fréchet Spaces with Basis. *Let X be a countable set, and F a Fréchet space with absolute basis $\{e_\alpha\}_{\alpha \in X}$. Assume there exists a continuous norm $\|\cdot\|_{00}$ on F for which $\|e_\alpha\|_{00} = 1$, $\alpha \in X$. (This happens, for example, if F is a dense Fréchet subspace of a C^* -algebra, with continuous inclusion, and each e_α is a partial isometry.) Let $\{\|\cdot\|_n\}_{n=0}^\infty$ be an increasing family of norms dominating $\|\cdot\|_{00}$ and topologizing F . Then $\sigma_n(\alpha) = \|e_\alpha\|_n$ defines a family of scales on X , and the Fréchet spaces $F \cong \mathcal{S}_\sigma^1(X)$ are naturally isomorphic.*

Moreover, F is nuclear if and only if σ satisfies the summability condition

$$(\forall n \in \mathbb{N}) (\exists m > n) \quad \sum_{\alpha \in X} \frac{\sigma_n(\alpha)}{\sigma_m(\alpha)} < \infty, \quad (6)$$

and if and only if the Fréchet spaces $F \cong \mathcal{S}_\sigma^1(X) \cong \mathcal{S}_\sigma^\infty(X)$ are naturally isomorphic.

Proof: The isomorphism $F \cong \mathcal{S}_\sigma^1(X)$ is [Pietsch, 1972], 10.1.4 Theorem. We recall the proof here. Define a linear map $\theta: \text{span}\{e_\alpha \mid \alpha \in X\} \rightarrow c_f(X) \hookrightarrow \mathcal{S}_\sigma^1(X)$ by $\theta(e_\alpha) = \delta_\alpha$.

For $f \in \text{span}\{e_\alpha \mid \alpha \in X\}$, we have

$$\begin{aligned}\|\theta(f)\|_n^1 &= \sum_{\alpha \in X} \sigma_n(\alpha) |\theta(f)(\alpha)| \\ &= \sum_{\alpha \in X} \|e_\alpha\|_n |f_\alpha| \leq C \|f\|_q,\end{aligned}\tag{7}$$

since the basis $\{e_\alpha\}_{\alpha \in X}$ is equicontinuous and absolute. So θ extends to a continuous map $\theta: F \rightarrow \mathcal{S}_\sigma^1(X)$. Since $\sum_{\alpha \in X} \|e_\alpha\|_n |f_\alpha| = 0$ implies $f = 0$, θ is injective (apply the first two lines of (7) for an arbitrary $f \in F$.)

Similarly define $\beta: c_f(X) \rightarrow \text{span}\{e_\alpha \mid \alpha \in X\}$ by $\beta(\delta_\alpha) = e_\alpha$. For $\varphi \in c_f(X)$, we have

$$\begin{aligned}\|\beta(\varphi)\|_n &= \left\| \sum_{\alpha \in X} \beta(\varphi)_\alpha e_\alpha \right\|_n \\ &= \left\| \sum_{\alpha \in X} \varphi(\alpha) e_\alpha \right\|_n \\ &\leq \sum_{\alpha \in X} |\varphi(\alpha)| \|e_\alpha\|_n = \|\varphi\|_n^1,\end{aligned}\tag{8}$$

so β extends to a continuous map $\beta: \mathcal{S}_\sigma^1(X) \rightarrow F$. It is easy to verify $\beta = \theta^{-1}$, so we have proved $\theta: F \cong \mathcal{S}_\sigma^1(X)$ is an isomorphism of Fréchet spaces.

Assume that $F = \mathcal{S}_\sigma^1(X)$ is nuclear. For this part of the proof, assume we have an ordering of X , so $X = \mathbb{N}$. Let F_n denote the Banach space $\ell^1(\mathbb{N}, \sigma_n)$. Then $F_0 \supset F_1 \supset \dots \supset F_n \supset \dots$ and $F = \bigcap_{n \in \mathbb{N}} F_n = \varprojlim_n F_n$. The connecting maps $\iota: \ell^1(\mathbb{N}, \sigma_m) \hookrightarrow \ell^1(\mathbb{N}, \sigma_n)$ may be taken to be type ℓ^1 by [Pietsch, 1972], 8.6.1 Theorem. This means that $\sum_{r=0}^\infty \alpha_r(\iota) < \infty$, where

$$\alpha_r(\iota) = \inf \{ \|\iota - T\|_{n,m} \mid T: F_m \rightarrow F_n, \dim(T(F_m)) \leq r \}.$$

Writing $T(f) = \sum_{i=1}^r (\sum_j c_{jk_i} f_j) \delta_{k_i}$, $f \in F$, where $c_{jk} \in \mathbb{C}$, we see that the norm

$$\|(\iota - T)(f)\|_n = \sum_{i=1}^r \left| f_{k_i} - \sum_j c_{jk_i} f_j \right| \sigma_n(k_i) + \sum_{k \neq k_1, \dots, k_r} |f_k| \sigma_n(k)$$

is minimized, simultaneously for every $f \in F$, if T is diagonal with $c_{k_i k_i} = 1$. For this T ,

$$\|\iota - T\|_{n,m} = \inf \left\{ \left\| \frac{\sigma_n}{\sigma_m} \upharpoonright_{k \neq k_1, \dots, k_r} \right\|_\infty \mid k_1, \dots, k_r \in \mathbb{N} \right\}. \quad (9)$$

Since $\alpha_r(\iota) \rightarrow 0$ as $r \rightarrow \infty$, it follows from (9) that $\sigma_n(k)/\sigma_m(k)$ must tend to 0 as $k \rightarrow \infty$.

So by reordering the basis (the k 's), we may assume $\sigma_n(0)/\sigma_m(0) \geq \sigma_n(1)/\sigma_m(1) \geq \dots \sigma_n(k)/\sigma_m(k) \geq \dots$, and (9) becomes $\|\iota - T\|_{n,m} = \sigma_n(r)/\sigma_m(r)$, with their common value equaling $\alpha_r(\iota)$. Since $\sum_r \alpha_r(\iota) < \infty$, the summability condition (6) follows.

For the remaining statements, see [Pietsch, 1972], §6.1, [Dubinsky, 1979], Chapter I, §3. or [Sch, 1993], Theorem 6.24, with G countable and discrete. \square

Definition 2.8. Regular Basis. A basis $\{e_k\}_{k \in \mathbb{N}}$ in a nuclear Fréchet space is *regular* if $\sigma_n(k)/\sigma_m(k) \geq \sigma_n(k+1)/\sigma_m(k+1)$ for all $m \geq n$ and $k \in \mathbb{N}$ [Dubinsky, 1979], Chapter I, (6.3.1), where σ is defined by $\sigma_n(k) = \|e_k\|_n$. Note that if σ is the power of a single scale $\sigma_n = \sigma^n$ on \mathbb{N} , regularity means σ is ordered so that $\sigma(0) \leq \sigma(1) \leq \sigma(2) \leq \dots$.

Proposition 2.9. The Summability Condition when $X = \mathbb{N}$. Define a scale on \mathbb{N} by $\sigma_{\text{id}}(k) = 1+k$, $k \in \mathbb{N}$, and let σ be any family of scales on \mathbb{N} . If $\sigma_{\text{id}} \lesssim \sigma$, then σ satisfies the summability condition (6). Conversely, if the regularity condition of Definition 2.8 is satisfied, and σ satisfies the summability condition (6), then $\sigma_{\text{id}} \lesssim \sigma$.

Proof: If $\sigma_{\text{id}} \lesssim \sigma$, then

$$\begin{aligned} \sum_{k \in \mathbb{N}} \frac{1}{\sigma_m(k)} &\leq \sum_{k \in \mathbb{N}} \frac{C}{\sigma_{\text{id}}(k)^2} && \text{since } \sigma_{\text{id}}(k)^2 \leq C\sigma_m(k) \\ &= \sum_{k \in \mathbb{N}} \frac{C}{(1+k)^2} = \frac{C\pi^2}{6} < \infty, \end{aligned} \quad (10)$$

so σ satisfies the summability condition.

For the converse, apply the Cauchy Condensation Test [Marsden, 1973] for convergence of the series $\sum_{k \in \mathbb{N}} \sigma_0(k)/\sigma_m(k)$, where $\sigma_0 \equiv 1$, to see that $\sum_{i \in \mathbb{N}} 2^i/\sigma_m(2^i) < \infty$. Then there exists a constant $C > 1$ so that $2^i \leq C\sigma_m(2^i)$, $i \in \mathbb{N}$. Next find $p > m$ so that $\sum_{k \in \mathbb{N}} \sigma_m(k)/\sigma_p(k) < \infty$, and apply the Cauchy Condensation Test again to see that $\sum_{i \in \mathbb{N}} 2^i\sigma_m(2^i)/\sigma_p(2^i) < \infty$. Let $D > 1$ be such that $2^i\sigma_m(2^i) \leq D\sigma_p(2^i)$. Combining with the previous result gives $2^{2i} \leq 2^i C\sigma_m(2^i) \leq CD\sigma_p(2^i)$.

By regularity $\sigma_p(k)$ increases with k , so $2^{2i} \leq CD\sigma_p(k)$ for $k \geq 2^i$. It follows that $k \leq CD\sigma_p(k)$ for $2^i \leq k \leq 2^{2i}$. Since for $i \geq 1$, $2^{i+1} \leq 2^{2i}$, we have $k \leq CD\sigma_p(k)$ for all $k \in \mathbb{N}$. Then $1+k \leq (1+CD)\sigma_p(k)$ shows $\sigma_{\text{id}} \lesssim \sigma$. \square

3 Fréchet Ideals

Definition 3.1. Fréchet Ideals and Continuous Inclusion. Let B be a Banach algebra, with norm $\|\cdot\|_B$, and A a subalgebra of B . The algebra A is called a *Fréchet algebra* when endowed with a Fréchet space topology for which multiplication is jointly continuous. Let $\{\|\cdot\|_n\}_{n=0}^\infty$ be an increasing family of seminorms giving the topology for A . When we say that A is a *Fréchet subalgebra* of B , we require the inclusion map

$A \hookrightarrow B$ be continuous. In terms of seminorms, this means that if $n \in \mathbb{N}$ is sufficiently large, there exists a constant $C > 0$ such that

$$\|a\|_B \leq C\|a\|_n, \quad (11)$$

for $a \in A$. Define new norms $\{\|\cdot\|'_n\}_{n=0}^\infty$ on A by $\|a\|'_0 = \|a\|_B$ and $\|a\|'_{n+1} = \max\{\|a\|_n, \|a\|_B\}$.

By continuous inclusion (11), the “primed” norms also topologize A , and we will use them in place of our original family.

We say that A is a *right Fréchet ideal* in B if $a \in A$ and $b \in B$ implies $ab \in A$, and this multiplication operation is continuous for the respective topologies. In terms of norms, this means that for each $n \in \mathbb{N}$ there exists an integer $m \geq n$ and constant $C_n > 0$ such that the inequality

$$\|ab\|_n \leq C_n \|a\|_m \|b\|_B \quad (12)$$

holds for all $a \in A$ and $b \in B$. Similarly, a *left Fréchet ideal* and *two-sided Fréchet ideal* is defined.

If $\{\|\cdot\|'_n\}_{n=0}^\infty$ is an equivalent increasing family of seminorms for A , then the inequality (12) is still satisfied but with adjusted constants C'_n and integers m'_n . If $\|\cdot\|'_B$ is an equivalent Banach algebra norm on B , then the constants C_n will scale uniformly in n , with each m_n staying the same for a given n in (12).

Remark 3.2. A subalgebra A of an algebra B is *spectral invariant* in B if every element $a \in A$ is quasi invertible in B if and only if it is quasi invertible in A . An element $x \in B$ is a *quasi-inverse* for $y \in B$ if $x \circ y = y \circ x = 0$, where $x \circ y$ is defined as $x + y + xy$ for any $x, y \in B$. If A is a left or right ideal in an algebra B , then A is spectral invariant

in B . For let $x \in A$ have quasi-inverse $y \in B$. If A is a right ideal, then $xy \in A$. So $0 = x \circ y = x + y + xy$ and $y = -x - xy \in A$. For left ideals, apply the same argument with yx in place of xy .

We say that a scale ℓ on \mathbb{N} is *proper* if the inverse map ℓ^{-1} takes bounded subsets of $[1, \infty)$ to finite subsets of \mathbb{N} .

Example 3.3. Let $B = c_0(\mathbb{N})$ be the commutative C^* -algebra of complex-valued sequences which vanish at infinity, with pointwise multiplication and sup-norm $\|\cdot\|_B = \|\cdot\|_\infty$.

(a) Let A be the dense Banach subalgebra $l^p(\mathbb{N})$ for some $1 \leq p < \infty$, with pointwise multiplication. The inequality $\|fg\|_p \leq \|f\|_p \|g\|_\infty$ is satisfied for all $f, g \in A$, so A is a dense Banach ideal in B . A is not nuclear by Proposition 2.1 (b).

(b) Let ℓ be a proper scale on \mathbb{N} , and let A be the Fréchet space of ℓ -rapidly vanishing sequences $\mathcal{S}_\ell^\infty(\mathbb{N})$, topologized by the sup-norms $\|f\|_n = \|\ell^n f\|_\infty$. The inequalities $\|fg\|_n \leq \|f\|_n \|g\|_\infty$ are all satisfied, so A is a dense Fréchet ideal in B . A is nuclear if there exists a $p \in \mathbb{N}^+$ for which $\sum_{k=0}^\infty \frac{1}{\ell(k)^p} < \infty$ (Theorem 2.7). For example, this sum is bounded with $p = 2$ for standard Schwartz functions on \mathbb{N} (where $\ell(k) = k + 1$ - see Equation (1)).

Proposition 3.4. Unital Banach Algebras. *Let A be a subalgebra of a Banach algebra B . If A is dense in B , then a (left or right) unit for A is also a (left or right) unit for B .*

If B has a two-sided unit or left (right) unit, and A is a dense right (left) ideal, then A contains the same unit. If B has a left (right) unit 1_B , and A is a dense left (right)

ideal, then A contains a left (right) unit, possibly different than 1_B .

If A and B have the same (left or right) unit, and A is a (right or left) ideal in B , then $A = B$. If A is a (right or left) Fréchet ideal, the equality $A = B$ is topological, and A is a Banach algebra exactly equal to B .

In other words, dense ideals can only be proper when the Banach algebra B is non-unital.

Remark 3.5. One-sided Units Not Unique. If B has a left unit 1_L , and A is a left ideal, then A will contain a left unit, but maybe different than 1_L . (A similar statement holds for right units.) Let $B = \ell^2(\mathbb{N})$ with multiplication $\chi * \eta = \chi_0 \eta$. Then $1_L = (1, 0, \dots, 0, \dots)$ is a natural left unit to take for B . Let $\xi = (1, 1/2, 1/3, \dots, 1/k, \dots) \in \ell^2(\mathbb{N})$, and take $A = \mathbb{C}\xi \oplus (\xi^\perp \cap c_f(\mathbb{N}))$. Note that A is a left ideal in B , but not a right ideal, and $\xi \in A$ is a left unit for B , but $1_L \notin A$. To see that A is dense in B , let $\chi \in \xi^\perp$ and $\epsilon > 0$. Let $\eta \in c_f(\mathbb{N})$ satisfy $\|\eta - \chi\|_2 < \epsilon$. Then $\eta' = \eta - \langle \eta, \xi \rangle \xi$ is in $\xi^\perp \cap c_f(\mathbb{N})$, and $\|\eta' - \chi\|_2 \leq \epsilon + |\langle \eta, \xi \rangle| = \epsilon + |\langle \eta - \chi, \xi \rangle| \leq \epsilon + \|\eta - \chi\|_2 \|\xi\|_2 \leq \epsilon + \epsilon \|\xi\|_2$, using the Cauchy-Schwartz inequality. To make A a nuclear left Fréchet ideal, replace $\xi^\perp \cap c_f(\mathbb{N})$ with $\xi^\perp \cap \mathcal{S}(\mathbb{N})$.

Proof of Proposition 3.4: Let A be a dense subalgebra of B , and 1_L a left unit in A , satisfying $1_L a = a$ for every $a \in A$. Let $b \in B$, and $a_\epsilon \in A$ satisfy $\|a_\epsilon - b\|_B < \epsilon$. Then $\|1_L b - b\|_B \leq \|1_L b - 1_L a_\epsilon\|_B + \|1_L a_\epsilon - b\|_B \leq \|1_L\|_B \epsilon + \epsilon$. Letting $\epsilon \rightarrow 0$, we see that $1_L b = b$ and 1_L is a left unit for B . The same argument works for a right unit.

Let 1_B be a two-sided unit in B , and let $a \in A$ be close to 1_B in $\|\cdot\|_B$, so that $b = a^{-1}$

exists in B . If A is a right ideal, then $1_B = ab \in A$, so A is unital with the same unit as B . The same argument holds if A is a left ideal.

Let $1_L \in B$ be a left unit for B . Then $A1_L$ is a dense subalgebra of the closed subalgebra $B1_L$ of B , and 1_L is a two-sided unit for $B1_L$, so $1_L \in A1_L$ by the previous paragraph. If A is a right ideal of B , then $A1_L \subseteq A$ so $1_L \in A$. If A is a left ideal, let $a_0 \in A$ be such that $1_L = a_0 1_L$. Then a_0 is a left unit for B contained in A . A similar argument applies to right units.

Next assume that A is a right ideal and $1_L \in A$ is a left unit for A and B . Then for every $b \in B$, $b = 1_L b \in A$, so $A = B$. If A is a right Fréchet ideal, then $\|a\|_n = \|1_L a\|_n \leq C_n \|1_L\|_m \|a\|_B = C'_n \|a\|_B$ for all $a \in A$, so each seminorm for A is bounded by $\|\cdot\|_B$. By continuous inclusion $A \hookrightarrow B$ (11), A has a topology equivalent to B 's. \square

Proposition 3.6. Quotients and Idempotent Subalgebras. *Let A be a Fréchet subalgebra of a Banach algebra B .*

(a) *If J is a closed two-sided ideal of B , then the image of A in the quotient Banach algebra B/J is a Fréchet subalgebra. The image of A in the quotient B/J is a left (right) Fréchet ideal if A is a left (right) Fréchet ideal in B . The image is nuclear if A is nuclear, and dense if A is dense in B .*

(b) *Let e be an idempotent in B . Then Be and eB are closed subalgebras of B . If A is a left (right) Fréchet ideal in B , then Ae (eA) inherits a natural quotient Fréchet topology from A , and is a left (right) Fréchet ideal in B . Both Ae and eA are nuclear if A is, and Ae (eA) is dense in Be (eB) if A is dense in B . If $e \in A$, then Ae and eA are closed subalgebras of A .*

Note that Remark 3.5 gives an example when $e \notin A$, and Ae is not contained in A .

Proof of (a): Let $\pi: B \longrightarrow B/J$ be the canonical quotient map. The Fréchet algebra A maps into B/J by a composition of continuous maps $\pi \circ \iota: A \rightarrow B/J$, where $\iota: A \hookrightarrow B$ is the inclusion map. So the kernel $I = J \cap A$ is a closed two-sided ideal in A , and we identify A/I with a Fréchet subalgebra of B/J .

Assume that A is a right Fréchet ideal in B , and let $a \in A$ and $b \in B$. Then $ab \in A$ so $\pi(a)\pi(b) \in A/I$. Let $\{\|\cdot\|_n\}_{n=0}^\infty$ be increasing seminorms for A , and $\|\cdot\|_B$ the norm on B . The n th quotient seminorm for A/I is

$$\|\pi(a)\|_n = \inf_{i \in I} \|a + i\|_n,$$

and similarly $\inf_{j \in J} \|b + j\|_B$ is the norm on B/J . Using the seminorm inequality (12) for right ideals,

$$\begin{aligned} \|\pi(a)\pi(b)\|_n &= \inf_{i \in I} \|ab + i\|_n \\ &= \inf_{i \in I} \inf_{j \in J} \|a(b + j) + i\|_n \quad \text{since } AJ \subseteq A \cap J = I \\ &\leq \inf_{i \in I} \inf_{j \in J} \|(a + i)(b + j)\|_n \quad \text{since } IB \subseteq A \cap J = I \\ &\leq \inf_{i \in I} \inf_{j \in J} C_n \|(a + i)\|_m \|(b + j)\|_B \quad \text{by inequality (12)} \\ &= C_n \|\pi(a)\|_m \|\pi(b)\|_B, \end{aligned} \tag{13}$$

so A/I is a right Fréchet ideal in B/J . The “left” case is handled in the same way.

If A is nuclear, then A/I is nuclear, since quotients by closed linear subspaces preserve nuclearity [Pietsch, 1972], Proposition 5.1.3, or [Treves, 1967], Proposition 50.1 (50.4). If $a_k \in A$ is a sequence approaching b in B , then by continuity $\pi(a_k)$ approaches $\pi(b)$ in

B/J . So A/I is dense in B/J whenever A is dense in B .

Proof of (b): Let e be an idempotent in B , and consider the linear subspace $Be = \{be \mid b \in B\}$ of B . If $\{b_k\}_{k \in \mathbb{N}}$ is a sequence in the Banach algebra B , such that $b_k e$ converges in B to a limit b_0 , then $b_0 e = b_0$, since multiplication is continuous. So Be is a closed linear subspace of B , and clearly a subalgebra. Similarly for eB . If $e \in A$, the same argument shows Ae and eA are closed subalgebras of the Fréchet algebra A .

Assume that A is a left Fréchet ideal in B . Then Ae is a left ideal in B since $b(ae) \in B(Ae) \subseteq (BA)e \subseteq Ae$, for $a \in A$ and $b \in B$. Note that $I = \{a \in A \mid ae = 0\}$ is a closed left ideal in A , with quotient $A/I = Ae$. The inherited topology on Ae is given by the quotient seminorms $\|ae\|'_n = \inf_{i \in I} \|a + i\|_n$, for $ae \in Ae$. For $b \in B$, $ae \in Ae$, we have

$$\begin{aligned}
\|b(ae)\|'_n &= \inf_{i \in I} \|ba + i\|_n \\
&\leq \inf_{i \in I} \|b(a + i)\|_n \\
&\leq C_n \|b\|_B \inf_{i \in I} \|a + i\|_m \\
&= C_n \|b\|_B \|ae\|'_m,
\end{aligned} \tag{14}$$

so Ae is a left Fréchet ideal in B .

If A is nuclear, then Ae is nuclear, since quotients by closed linear subspaces preserve nuclearity [Pietsch, 1972], Proposition 5.1.3, or [Treves, 1967], Proposition 50.1 (50.4). If $be \in B$ and $a_k \rightarrow b$ in B , then $a_k e \rightarrow be$ in Be , by continuity of multiplication, so Ae is dense in Be if A is dense in B . □

Remark 3.7. Note that the quotient Fréchet ideal $A/I \subseteq B/J$ in Proposition 3.6 (a) satisfies inequality (12) with the same integers m_n and constants C_n as the original Fréchet ideal $A \subseteq B$. The same is true for the ideals $Ae \subseteq B$ (or $eA \subseteq B$).

Proposition 3.8. Algebraic Ideals. *Let A be a Fréchet subspace of a Banach algebra B , with continuous inclusion $\iota: A \hookrightarrow B$. If A is a left (right) ideal in the purely algebraic sense, then A is a left (right) Fréchet ideal in B .*

Proof: Let $L_b: A \rightarrow A$ denote left multiplication by some $b \in B$. Assume $a_\alpha \rightarrow 0$ and $L_b(a_\alpha) \rightarrow a_0$ in A . Then $a_\alpha \rightarrow 0$ and $L_b(a_\alpha) \rightarrow a_0$ in B , by continuous inclusion. But since B is a Banach algebra, the multiplication is continuous, and $a_0 = 0$. Apply the closed graph theorem to see that L_b is a continuous linear map from A to A .

Let $R_a: B \rightarrow A$ denote right multiplication by some $a \in A$. Let $b_\alpha \rightarrow 0$ in B and $b_\alpha a \rightarrow a_0$ in A . Then $b_\alpha a \rightarrow a_0$ in B , by continuous inclusion, and so $a_0 = 0$. Apply the closed graph theorem again to see that R_a is continuous from B to A .

We have shown that the bilinear map of multiplication $M: B \times A \rightarrow A$ is separately continuous. By [Rudin, 1973], Theorem 2.17, M is jointly continuous. \square

4 Dense Nuclear Ideals in C^* -Algebras

In this section and the next, we apply the dense ideal condition together with nuclearity, to get results about the structure of the Banach algebra. Here we obtain complete results for C^* -algebras, but must wait until §5 to finish general Banach algebras.

An algebra is *semiprime* if it has no non-zero nilpotent ideal. (One can use two-sided, left, or right ideals to define semiprime; the resulting definitions are equivalent [Palmer, 1994], Proposition 4.4.2 (d).)

Proposition 4.1. *Let A be a dense nuclear left (right) Fréchet ideal of a Banach algebra B . Let $e \in B$ be an idempotent. Then Ae (eA) is a finite dimensional Banach algebra equal to Be (eB).*

Assume further that B is semiprime. Then Be , eB , and BeB are all finite dimensional Banach algebras equal to Ae , eA , and AeA respectively. The Fréchet ideal A can be either left or right for this to work.

Remark 4.2. If B is not semiprime, it may happen that A is a dense nuclear left Fréchet ideal, but eB and BeB are not finite dimensional, for some idempotent $e \in A$. Let B be the Hilbert space $l^2(\mathbb{N})$ and A be Schwartz functions $\mathcal{S}(\mathbb{N})$, with the natural inclusion map $A \hookrightarrow B$. For $f, g \in B$, define multiplication by $fg(n) = f(0)g(n)$. Then B is a Banach algebra and A is a Fréchet algebra for this multiplication. B is not semiprime since the (two-sided) ideal $I = \{f \in B \mid f(0) = 0\}$ satisfies $I^2 = 0$. The Fréchet algebra A is a dense nuclear left Fréchet ideal in B , but not a right ideal since $AB = B$. Let $e = (1, 0, 0, \dots) \in A$. Note that $e^2 = e$, $Be = \mathbb{C}e$, $eB = B$ and $BeB = B$.

Proof of Proposition 4.1: By Proposition 3.6 (b), Ae is a dense nuclear left Fréchet ideal in Be . Since e is a right unit for Be , Proposition 3.4 tells us $e \in Ae$. So by the last part of Proposition 3.4, Ae is a Banach algebra exactly equal to Be . Being a nuclear Banach space, $Ae = Be$ is finite dimensional (Proposition 2.1 (b)).

Next, I will imitate the proof of the first Lemma of [Smyth, 1980], to show that eB is also finite dimensional, with the added assumption that B is semiprime. We just proved that Be is finite dimensional, so eBe must also be finite dimensional. Assume the dimension is some positive integer N , and let $\theta: eBe \rightarrow \mathbb{C}^N$ be a linear bijective map. For each $y \in eB$, define a linear map $\varphi_y: Be \rightarrow \mathbb{C}^N$ by $\varphi_y(x) = \theta(yx)$, $x \in eB$. The map $y \mapsto \varphi_y$ from eB to the (finite dimensional) space of linear maps $\mathcal{L}(Be, \mathbb{C}^N)$ is linear. If $\varphi_{y_0} = 0$ for some $y_0 \in eB$, then $y_0Be = 0$, and $(By_0)^2 = (Bey_0)^2$ (since $y_0 \in eB$) $= Be(y_0Be)y_0 = 0$. So the left ideal By_0 is nilpotent with order 2, contradicting our assumption that B is semiprime. Therefore the mapping $y \in eB \mapsto \varphi_y \in \mathcal{L}(Be, \mathbb{C}^N)$ is one-to-one, and hence $\dim(eB) < \dim(\mathcal{L}(Be, \mathbb{C}^N)) = \dim(Be) * N < \infty$.

Now we know that both eB and Be are finite dimensional. Let $b_1, \dots, b_K \in B$ and $c_1, \dots, c_L \in B$ satisfy $eB = \mathbb{C}\text{-span}\{eb_i \mid i = 1, \dots, K\}$ and $Be = \mathbb{C}\text{-span}\{c_ie \mid i = 1, \dots, L\}$. Then BeB has dimension at most KL since $BeB = \mathbb{C}\text{-span}\{biec_j \mid i = 1, \dots, K, \quad j = 1, \dots, L\}$. □

Theorem 4.3. *Let B be a commutative C^* -algebra, with maximal ideal space M . Then B has a dense nuclear Fréchet ideal if and only if M is discrete and countable.*

Proof: The maximal ideal space M is locally compact, and B is isomorphic to the C^* -algebra $C_0(M)$ of continuous functions vanishing at infinity on M [Dixmier, 1982], §1.4.1. Let $m \in M$, and let $U \subseteq M$ be an open, relatively compact set about m . The set of functions $f \in B$ which vanish on the closure \overline{U} , is a closed ideal $I_{\overline{U}}$ in B . Let A be a dense nuclear Fréchet ideal in B . Applying Proposition 3.6 (a), we find the image $\pi(A)$ of A in the quotient $B/I_{\overline{U}}$ is again a dense nuclear Fréchet ideal. Since $B/I_{\overline{U}} \cong C(\overline{U})$

is unital by the compactness of \overline{U} , Proposition 4.1 (with $e = 1_{C(\overline{U})}$) tells us that $\pi(A)$ is equal to $B/I_{\overline{U}}$ and finite dimensional. But $C(\overline{U})$ can only be finite dimensional if \overline{U} is a finite set of points. Thus every point $m \in M$ has a finite neighborhood. This proves that M is discrete.

To see that M is countable, let δ_m denote the unit step function at $m \in M$. An element $b \in B$ can satisfy $\|b - \delta_m\|_B < 1/4$ for at most one $m \in M$, since $\|\delta_{m_1} - \delta_{m_2}\|_B = 1$ for distinct $m_1, m_2 \in M$. By Corollary 2.3, B has a countable dense set S . The correspondence $m \mapsto$ “choose $b \in S$ within distance $1/4$ of δ_m ” gives an injective map from M into S , so M is countable.

Since M is discrete and countable, it must be either finite or isomorphic to the natural numbers \mathbb{N} . For the infinite case, $B \cong c_0(\mathbb{N})$, and the standard set of Schwartz functions $A = \mathcal{S}(\mathbb{N})$ is a dense nuclear Fréchet ideal (see Equation (1) or Example 3.3 (b)). \square

Lemma 4.4. Existence of Projections. *Let B be a C^* -algebra with a dense nuclear left or right Fréchet ideal A . Let I be any proper closed two-sided ideal in B . Then A contains a nontrivial projection which does not lie in I .*

Proof: Let a be an element of $B - I$, with image $[a]$ in the quotient B/I . Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be an approximate identity for the ideal I . Assume A is a right Fréchet ideal, and note that $\|a - ae_\lambda\|_B$ approaches $\|[a]\|_{B/I}$. By the Fréchet ideal condition, $\|ae_\lambda\|_n \leq C_n \|a\|_m \|e_\lambda\|_B = C_n \|a\|_m$, $n \in \mathbb{N}$, $\lambda \in \Lambda$. Since a bounded set in a nuclear locally convex space is relatively compact (Proposition 2.1 (d)), there is a cluster point $i_0 \in I \cap A$ for the net $\{ae_\lambda\}_{\lambda \in \Lambda}$, converging in the Fréchet topology of A . Replace a with $a - i_0$ so now we have $\|a\|_B = \|[a]\|_{B/I} \neq 0$. Replace a with $aa^*/\|a\|_B^2$. This new a remains in A , is

positive, and satisfies $\|a\|_B = \|[a]\|_{B/I} = 1$.

For each $k \in \mathbb{N}$, a^{2^k} has C^* -norm equal to one, by applying the C^* -identity $\|a^{2^{k+1}}\|_B = \|a^{2^k}\|_B^2$ repeatedly. By the Fréchet ideal condition, $\|a^{2^k}\|_n \leq C_n \|a^{2^{k-1}}\|_B \|a\|_m \leq C_n \|a\|_m$, $k, n \in \mathbb{N}$. Since a bounded set in a nuclear locally convex space is relatively compact (Proposition 2.1 (d)), there is a subsequence $\{a^{k_i}\}_{i=0}^\infty$, converging in the Fréchet topology of A . The limit point, $a_0 \in A$, must also have unit norm in B . Moreover, the image $[a_0]$ of a_0 in B/I also has norm 1, since $\|[a^{k_i}]\|_{B/I} = 1$ for each k_i .

Consider the commutative C^* -subalgebra $C^*(a)$ of B generated by a . This algebra must be isomorphic to continuous functions on a locally compact space M , vanishing at infinity. We may think of a and a_0 as real-valued functions on M , with range in $[0, 1]$, both taking on the value 1 for at least one point of M . Since $a^{k_i} \rightarrow a_0$ in the sup-norm, $a^{k_i}(m) \rightarrow a_0(m)$ for each $m \in M$. It follows that $a_0(m) \in \{0, 1\}$, and a_0 is a projection. \square

Definition 4.5. The Finite Socle. Let A be any algebra. The *left (right) finite socle of A* is the sum of all minimal left (right) finite dimensional ideals of A . If the left and right finite socles are equal, their common value is the *finite socle of A* , denoted by A_{fin} .

For $a \in A$ and minimal left ideal L of A , La is a minimal left ideal or $\{0\}$ [Palmer, 1994], Proposition 8.2.8. If L is finite dimensional, so is La . Hence the left finite socle is a right, and therefore two-sided, ideal of A . Similarly the right finite socle is a two-sided ideal of A .

Let S be the set $\{e \mid e \text{ is a minimal idempotent in } A\}$, and assume A is semiprime. Then $\{Ae \mid e \in S\}$ ($\{eA \mid e \in S\}$) gives all the minimal left (right) ideals of A [Palmer,

1994], Corollary 8.2.3. By the first Lemma of [Smyth, 1980], we know that $\dim(Ae)$ is finite if and only if $\dim(eA)$ is finite. Let S_{fin} denotes those $e \in S$ for which Ae (or eA) is finite dimensional. Then the left (right) finite socle is equal to AS_{fin} ($S_{\text{fin}}A$). By the previous paragraph, these are both two-sided ideals in A . But then they must both equal $AS_{\text{fin}}A$, so the left and right finite socles agree, and $A_{\text{fin}} = AS_{\text{fin}} = S_{\text{fin}}A = AS_{\text{fin}}A$. Note that C^* -algebras are semiprime.

Theorem 4.6. *Let B be a C^* -algebra containing a dense nuclear left or right Fréchet ideal A . Then the finite socle B_{fin} is dense in both B and A , the primitive ideal space $\text{Prim}(B)$ is discrete and countable, and B/I is finite dimensional for any $I \in \text{Prim}(B)$. B is the countable direct sum, or restricted product, of finite dimensional matrix algebras.*

Proof: We do the proof for a left ideal A . The closure $I = \overline{B_{\text{fin}}}^B$ is a two-sided ideal in B . Assume for a contradiction that $I \neq B$. Apply Lemma 4.4 to get a nontrivial projection $p \in A - I$. By Proposition 4.1, $Ap = Bp$ is finite dimensional. If $I \cap Bp \neq \{0\}$, find only finitely many $e_1, \dots, e_k \in S_{\text{fin}}$ such that $e_i p \neq 0$. A simple calculation shows $e' = p - e_1 - \dots - e_k$ is an idempotent in B which is orthogonal to I . Since $p \notin I$, $e' \neq 0$. By Proposition 4.1, $Ae' = Be'$ is finite dimensional. This is a left ideal of B whose intersection with I is $\{0\}$. Let e_{min} be a minimal idempotent contained therein. Then Be_{min} is a minimal left finite dimensional ideal in B , which is not in B_{fin} , a contradiction.

Let e be any idempotent in B . The second part of Proposition 4.1 tells us $eA = eB$. Since $e \in eB$, we know $e \in eA$. But $eA \subseteq A$ since A is a left ideal. So A contains any idempotent of B , $S_{\text{fin}} \subset A$, and $B_{\text{fin}} \subseteq A$.

The distance in B between two idempotents $e, f \in S_{\text{fin}}$ is at least one, since $\|e -$

$f\|_B\|e\|_B \geq \|e - ef\|_B = \|e\|_B \geq 1$. By Corollary 2.3, B is separable, so S_{fin} is at most countable.

For $e \in S_{\text{fin}}$, BeB is a finite dimensional matrix algebra with dimension $(\dim Be)^2$. So as a C^* -algebra, B is the direct sum of finite dimensional matrix algebras, and $\text{Prim}(B)$ is well-known to be discrete [Fell Dor, 1988], Proposition 5.21. Also B/I is one of the finite dimensional direct summands for each $I \in \text{Prim}(B)$.

To see that B_{fin} is dense in A , let $P_K = \sum_{k \leq K} 1_k$ be the sum of the first K identity matrices, from the matrix algebras in the direct sum for B . Then $\{P_K\}_{K=0}^\infty$ is a bounded approximate unit for B . For any $a \in A$, the left ideal condition tells us that the set $\{P_K a \mid K \in \mathbb{N}\}$ is bounded in A . By Proposition 2.1 (d), some $a_0 \in A$ is a cluster point. But a_0 is also a cluster point in the topology of B , so since $P_K a \rightarrow a$ in B , we must have $a_0 = a$. Since $P_K a \in B_{\text{fin}}$, this shows that B_{fin} is dense in A . \square

There is a notion of nuclearity for C^* -algebras [Kad Ring II, 1997], Chapter 11, which is different from the notion of nuclearity for locally convex spaces.

Corollary 4.7. Nuclearity of the C^* -algebra. *Let B be a C^* -algebra with a dense left or right Fréchet ideal A . If A is nuclear as a locally convex space, then B is nuclear as a C^* -algebra.*

Proof: By Theorem 4.6, every irreducible representation of B is finite dimensional, and B is separable by Corollary 2.3. Applying [Dixmier, 1982], §9.1 Theorem (iv) \Rightarrow (i), we find that B is a Type I C^* -algebra. But every Type I C^* -algebra is nuclear in the C^* -algebraic sense [Paterson, 1988], (1.31). \square

5 Complete Continuity and the General Banach Case

In this section we generalize Theorem 4.6 for an arbitrary Banach algebra.

For any Banach algebra B , we give $\text{Prim}(B)$ the Jacobson topology [Dixmier, 1982], §3.1.1. Our main goal in this section is to show that $\text{Prim}(B)$ has a discrete topology, whenever B has a dense nuclear Fréchet ideal. The proof of Theorem 4.3 does not easily generalize to the noncommutative case, and the proofs of Theorem 4.6 and Lemma 4.4 rely on the theory of C^* -algebras. In this section, we find a different approach to the problem, with the concept of complete continuity.

If X and Y are Banach spaces, a continuous linear map $T: X \rightarrow Y$ is *completely continuous* if it maps weakly converging sequences in X to norm converging sequences in Y . A Banach algebra B is said to be *left completely continuous* if for every $b \in B$, the left multiplication operator $L_b: B \rightarrow B$, given by $x \in B \mapsto bx$, is a completely continuous map from B to itself. Similarly, B is *right completely continuous* if right multiplication R_b is completely continuous for every $b \in B$, and B is *completely continuous* if it is both left and right completely continuous [Kaplansky, 1949].

Theorem 5.1. *Let B be a Banach algebra containing a dense nuclear left (right) Fréchet ideal A . Then B is right (left) completely continuous.*

Proof: We do the proof for right Fréchet ideals. Let $\{x_k\}_{k=0}^\infty$ be a sequence in B , and a a fixed element of A . It is not hard to show that if x_k converges to a limit x in the norm of B , then ax_k converges to ax in the Fréchet topology of A , using the right ideal inequality (12). But we must begin by assuming x_k converges weakly, *not* in norm.

Let φ be a continuous linear functional on A . Then $\varphi \circ L_a$ is continuous on B , since

$$\begin{aligned} |\varphi \circ L_a(x)| = |\varphi(ax)| &\leq C\|ax\|_n \quad C, n \text{ exist since } \varphi \text{ is continuous on } A \\ &\leq CC_n\|a\|_m\|x\|_B \quad \text{by the ideal inequality (12),} \end{aligned} \quad (15)$$

for any $x \in B$. Thus if our original sequence $\{x_k\}_{k=0}^\infty$ converges weakly to zero in B , then ax_k converges weakly to zero in A . To simplify notation, define $y_k = ax_k$. We wish to prove that $y_k \rightarrow 0$ in the norm of B , by using the fact that $y_k \rightarrow 0$ weakly in A .

Let U be the unit ball of B . Then $U \cap A$ is absolutely convex, and by continuous inclusion $A \hookrightarrow B$, $U \cap A$ is a zero neighborhood of A . Apply nuclearity [Pietsch, 1972], Proposition 4.1.4, to find an absolutely convex zero neighborhood V of A , and sequence of continuous linear functionals $\varphi_n \in A'$ satisfying

$$\sum_{n=0}^{\infty} \|\varphi_n\|_{V^0} < \infty, \quad (16)$$

such that the inequality $\|x\|_B \leq \sum_{n=0}^{\infty} |\varphi_n(x)|$ holds for all $x \in A$. Taking $x = y_k$, we get

$$\|y_k\|_B \leq \sum_{n=0}^{\infty} |\varphi_n(y_k)| \quad (17)$$

holds for every $k \in \mathbb{N}$.

Lemma 5.2. *The sequence $\{y_k\}_{k=0}^\infty$ is bounded in the seminorm $\|\cdot\|_V$ on A .*

Proof: Since $y_k \rightarrow 0$ weakly in the Fréchet space A , the set $\{y_k\}_{k=0}^\infty$ is weakly bounded in A . In the terminology of [Rudin, 1973], Theorem 3.18, V is an original neighborhood of A . So $\{y_k\}_{k=0}^\infty$ is contained in tV for some $t > 0$. \square

We use Lemma 5.2 to show that the right hand side of inequality (17) tends to zero

as $k \rightarrow \infty$. Let $\epsilon > 0$. Let N be large enough so that $\sum_{n=N+1}^{\infty} \|\varphi_n\|_{V^0} < \epsilon/2M$, where M is the bound on $\|y_k\|_V$ from Lemma 5.2. This is possible since the full series in (16) converges. Using weak convergence of $\{y_k\}_{k=0}^{\infty}$ to zero, find K large enough so that $\sum_{n=0}^N |\varphi_n(y_k)| < \epsilon/2$ for $k \geq K$. So we have

$$\begin{aligned}
\sum_{n=0}^{\infty} |\varphi_n(y_k)| &\leq \sum_{n=0}^N |\varphi_n(y_k)| + \sum_{n=N+1}^{\infty} |\varphi_n(y_k)| \\
&\leq \epsilon/2 + \sum_{n=N+1}^{\infty} |\varphi_n(y_k)| \quad \text{since } k \geq K \\
&\leq \epsilon/2 + \sum_{n=N+1}^{\infty} \|\varphi_n\|_{V^0} M \quad \text{since } \|y_k\|_V \leq M \text{ by Lemma 5.2} \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned} \tag{18}$$

Thus the right hand side of inequality (17) is less than ϵ for $k \geq K$, so the left hand side $\|y_k\|_B$ is also less than ϵ for $k \geq K$. Thus $y_k \rightarrow 0$ in the norm of B .

We have proved that if $\{x_k\}_{k=0}^{\infty}$ is a sequence in B converging weakly to zero, and $a \in A$, then $y_k = ax_k$ converges to zero in the norm of B . To finish Theorem 5.1 and prove the complete continuity of B , we would like to replace $a \in A$ with an arbitrary element b of B . Since $x_k \rightarrow 0$ weakly in B , the Uniform Boundedness Principle (Theorem 2.4), gives us an $M < \infty$ for which $\|x_k\|_B \leq M$ for all $k \in \mathbb{N}$. Let $\epsilon > 0$ and pick $a \in A$ such that $\|a - b\|_B < \epsilon/2M$. Let K be big enough so that $\|ax_k\|_B < \epsilon/2$ for $k \geq K$. Then

$$\begin{aligned}
\|bx_k\|_B &\leq \|(a - b)x_k\|_B + \|ax_k\|_B \\
&\leq \|a - b\|_B \|x_k\|_B + \epsilon/2 \quad \text{since } k \geq K \\
&\leq (\epsilon/2M)M + \epsilon/2 = \epsilon,
\end{aligned} \tag{19}$$

so $bx_k \rightarrow 0$ in the norm of B . This completes the proof of Theorem 5.1. \square

A *primitive ideal* of a Banach algebra B is the kernel of a non-zero algebraically irreducible, continuous left Banach-space representation of B . (This is equivalent to the purely algebraic definition of primitive ideal [Palmer, 1994], Corollary 4.2.9.) The *primitive ideal space* of B , or $\text{Prim}(B)$, is the set of all primitive ideals in B . We also call $\text{Prim}(B)$ the *spectrum* of B .

Corollary 5.3. *Let B be a Banach algebra containing a dense nuclear two-sided Fréchet ideal A . Then B is completely continuous and the primitive ideal space of B is discrete.*

Proof: By Theorem 5.1 just proved, containment of a dense nuclear two-sided Fréchet ideal implies B is completely continuous on both sides. By [Kaplansky, 1949], Theorem 5.1, a completely continuous Banach algebra has discrete primitive ideal space. \square

Proposition 5.4. *Let B be a Banach algebra with a dense nuclear left or right Fréchet ideal. Then B/I is finite dimensional for each $I \in \text{Prim}(B)$.*

Proof: Let V be an algebraically irreducible left Banach B -module. Then $I = \{b \in B \mid bV = 0\} \in \text{Prim}(B)$ is the primitive ideal corresponding to V . Let v be a nonzero element of V , and define $M = \{b \in B \mid bv = 0\}$. Then M is a maximal modular closed left ideal of B , and $V = B/M$ as left B -modules.

Let A be a dense nuclear left Fréchet ideal of B . Then Av is dense in V (since $Bv = V$ and A is dense in B), so there must be an $a_0 \in A$ such that $a_0v \neq 0$. Since A is a left ideal in B , we have $Av \supseteq (Ba_0)v = B(a_0v) = V$. Let $N = M \cap A = \{a \in A \mid av = 0\}$. Then N is a closed left ideal in A , $V = A/N$ as left B -modules, and A/N is nuclear [Pietsch, 1972], Proposition 5.1.3, or [Treves, 1967], Proposition 50.1 (50.4).

The map $a \in A \mapsto av \in V$ is continuous, so the Fréchet topology on A/N is at least

as strong as the original Banach space topology on V .² By [Treves, 1967], Chapter 17-7, Corollary 2, the topologies must agree. So V is a nuclear Banach space and therefore finite dimensional (Proposition 2.1 (b)). Since B/I is represented faithfully on V , it must also be finite dimensional. \square

Remark 5.5. An alternative proof of Proposition 5.4 (assuming a dense nuclear two-sided ideal) is given by applying the complete continuity of B from Theorem 5.1, and [Kaplansky, 1948], Lemma 4.

The *Jacobson radical* \mathcal{A}_J of an algebra \mathcal{A} is the intersection of all primitive ideals of \mathcal{A} . \mathcal{A} is *semisimple* if $\mathcal{A}_J = 0$, and *radical* if $\mathcal{A}_J = \mathcal{A}$ [Palmer, 1994], §4.3.1. A semisimple algebra is also semiprime [Palmer, 1994], Theorems 4.4.6 and 4.5.9.

Theorem 5.6. *Let B be a Banach algebra containing a dense nuclear two-sided Fréchet ideal A . Then $\text{Prim}(B)$ is discrete and countable, and B/I is finite dimensional for each $I \in \text{Prim}(B)$.*

Proof: Proposition 3.6 (a) shows the quotient of B by its Jacobson radical still has a dense nuclear Fréchet ideal. Since the Jacobson radical B_J is the intersection of all kernels of primitive ideals, $\text{Prim}(B) = \text{Prim}(B/B_J)$. So without loss of generality, I will assume that B is semisimple.

I will construct an idempotent for each primitive ideal. Let $I \in \text{Prim}(B)$ and let $\{I\}^c$ denote the complement of the singleton $\{I\}$, namely $\{J \in \text{Prim}(B) | J \neq I\}$. Discreteness (Corollary 5.3) tells us the intersection L of all elements of $\{I\}^c$ is not contained in I .

²The same is true if we give V the Banach space topology of B/M .

(The closure of $\{I\}^c$ in the Jacobson topology is by definition the set of primitive ideals containing L . By discreteness this closure must only be $\{I\}^c$ and nothing more.) Note that L , being the intersection of closed two-sided ideals, is itself a closed two-sided ideal in B . Intersecting L with I gives the zero ideal, since $L \cap I = \bigcap \{J \mid J \in \text{Prim}(B)\}$ and B is semisimple. So $LI = IL \subseteq L \cap I = 0$, and B is the direct sum of ideals $B = L \oplus I$. Since $\text{Prim}(B)$ is discrete, the singleton $\{I\}$ is a closed set, so I is not contained in any other primitive ideal. This implies L must be simple.

We need a unit for L . By Proposition 5.4, L is finite dimensional, and L is semiprime since B is [Palmer, 1994], Proposition 4.4.2 (e). By the Wedderburn Theorem [Palmer, 1994], Theorem 8.1.1, a finite dimensional semiprime algebra which is simple is isomorphic to a full matrix algebra, and therefore unital. Let e_I be the unit of L .

Since $LI = IL = 0$, we know $e_I I = I e_I = 0$. Let $J \in \text{Prim}(B)$, $J \neq I$, with corresponding idempotent e_J , satisfying $e_J J = J e_J = 0$. Since the singleton $\{J\}$ is closed in $\text{Prim}(B)$, J cannot be contained in I , so $L \cap J \neq 0$. Since $L \cap J$ is a two-sided ideal, and L is simple, we have $L \subseteq J$, so $e_I \in J$. It follows that $e_I e_J = e_J e_I = 0$. The distance between these two orthogonal idempotents is at least one, because $\|e_I - e_J\|_B \|e_I\|_B \geq \|e_I - e_J e_I\|_B = \|e_I\|_B$.

An element $b \in B$ can satisfy $\|b - e_I\|_B < 1/4$ for at most one $I \in \text{Prim}(B)$, since $\|e_I - e_J\|_B = 1$ for distinct $I, J \in \text{Prim}(B)$. By Corollary 2.3, B has a countable dense set S . The correspondence $I \mapsto$ “choose $b \in S$ within distance $1/4$ of e_I ” gives an injective map from $\text{Prim}(B)$ into S , so $\text{Prim}(B)$ is countable.

The last statement follows from Proposition 5.4. □

6 Construction of Dense Nuclear Ideals for C^* -Algebras

Assume that a C^* -algebra B is a countable direct sum of finite dimensional matrix algebras. Let $\mathbf{p} = \{p_k\}_{k=0}^\infty$ be the sequence of dimensions, so that $B = \bigoplus_{k=0}^\infty M_{p_k}(\mathbb{C})$.³ We think of elements of B as matrix-valued functions f on \mathbb{N} , where $f(k) \in M_{p_k}(\mathbb{C})$, $k \in \mathbb{N}$. The C^* -norm on B is

$$\|f\|_B = \sup_{k=0}^\infty \|f(k)\|_{\text{op}}, \quad (20)$$

and B consists of those functions f which vanish at ∞ [Dixmier, 1982], §1.9.14. In this section, we construct dense nuclear ideals in B .

Let X be the countable disjoint union of finite sets $\bigcup_{k=0}^\infty \{k\} \times \{0, \dots, p_k - 1\} \times \{0, \dots, p_k - 1\}$, and let $e_\alpha = e_{k,ij}$ be matrix elements for the $p_k \times p_k$ matrices $M_{p_k}(\mathbb{C})$, for each tuple $\alpha = \{k, i, j\} \in X$. These are partial isometries which form a Schauder basis for the C^* -algebra B (Definition 2.5). Any $b \in B$ has a coordinate functional $b_\alpha = \langle e_{k,ii} b e_{k,jj}, e_\alpha \rangle \in \mathbb{C}$, and unique series expansion $b = \sum_{\alpha \in X} b_\alpha e_\alpha$ which converges in B , since elements of B vanish at infinity.

Let $c_f(X)$ be the linear span of the e_α 's. The finite socle B_{fin} of B (Definition 4.5) is identified with $c_f(X)$, and equals the algebraic direct sum of the matrix algebras $\bigoplus_{k=0}^\infty M_{p_k}(\mathbb{C})$.

Definition 6.1. Socle-Specific Schwartz Spaces. Let ℓ be any family of scales on \mathbb{N} , with X defined above. The Fréchet space $\mathcal{S}_\ell^{\infty, \text{op}}(X)$ is defined to be the completion of

³We assume $p_k \geq 1$ for all k , unless the sequence \mathbf{p} terminates, in which case we assume there is a k_0 such that $p_k \geq 1$ for $k \leq k_0$, and $p_k = 0$ for $k > k_0$.

$c_f(X)$ in the norms

$$\|\varphi\|_n^{\infty, \text{op}} = \sup_{k \in \mathbb{N}} \ell_n(k) \|\varphi(k)\|_{\text{op}}, \quad (21)$$

where $\|\cdot\|_{\text{op}}$ is the operator norm on $M_{p_k}(\mathbb{C})$.

Theorem 6.2. *Let a C^* -algebra B be the countable direct sum of finite dimensional C^* -algebras, with sequence of dimensions \mathbf{p} . If ℓ is any family of scales on \mathbb{N} , then $\mathcal{S}_\ell^{\infty, \text{op}}(X)$ is a dense two-sided Fréchet ideal in B , in which $\{e_\alpha\}_{\alpha \in X}$ is an equicontinuous basis.*

Define a family of scales σ on X by $\sigma_n(k, i, j) = \ell_n(k)$ for $k, i, j \in X$, $n \in \mathbb{N}$. The Fréchet ideal is nuclear if and only if ℓ satisfies the \mathbf{p} -summability condition

$$(\forall n \in \mathbb{N}) (\exists m > n) \sum_{k \in \mathbb{N}} \frac{p_k^2 \ell_n(k)}{\ell_m(k)} < \infty, \quad (22)$$

and if and only if $\mathcal{S}_\ell^{\infty, \text{op}}(X) \cong \mathcal{S}_\sigma^1(X) \cong \mathcal{S}_\sigma^\infty(X)$.

It follows that a nuclear Fréchet ideal always exists, for any C^* -algebra B satisfying the hypotheses of Theorem 6.2, since ℓ_{sum} , defined in Definition 6.3, gives at least one family of scales satisfying (22).

Proof of Theorem 6.2: Since $\ell \geq 1$, $\|\cdot\|_0^{\infty, \text{op}} \geq \|\cdot\|_B$, and the inclusion map $\mathcal{S}_\ell^{\infty, \text{op}}(X) \hookrightarrow B$ is continuous. For $f \in B$ and $\varphi \in \mathcal{S}_\sigma^{\infty, \text{op}}(X)$, we have

$$\begin{aligned} \|f\varphi\|_n^{\infty, \text{op}} &= \sup_{k \in \mathbb{N}} \ell_n(k) \|f(k) * \varphi(k)\|_{\text{op}} && \text{definition of } \|\cdot\|_n^{\infty, \text{op}} \\ &\leq \left(\sup_{k \in \mathbb{N}} \|f(k)\|_{\text{op}} \right) \left(\sup_{k \in \mathbb{N}} \ell_n(k) \|\varphi(k)\|_{\text{op}} \right) \\ &= \|f\|_B \|\varphi\|_n^{\infty, \text{op}}, \end{aligned} \quad (23)$$

so $\mathcal{S}_\ell^{\infty, \text{op}}(X)$ is a left Fréchet ideal in B . Similarly it is a right, and therefore two-sided, Fréchet ideal in B .

The basis is equicontinuous since for $\varphi \in \mathcal{S}_\ell^{\infty, \text{op}}(X)$, $\alpha = (k, i, j) \in X$, and $k, n \in \mathbb{N}$,

$$|\varphi(k)_{ij}| \|e_\alpha\|_n^{\infty, \text{op}} = \|\varphi(k)_{ij} e_\alpha\|_n^{\infty, \text{op}} = \ell_n(k) \|e_{k,ii} \varphi(k) e_{k,jj}\|_{\text{op}} \leq \ell_n(k) \|\varphi(k)\|_{\text{op}} \leq \|\varphi\|_n^{\infty, \text{op}}.$$

Since the operator norm on $M_{p_k}(\mathbb{C})$ is bounded by the sum of the matrix entries, and is greater than or equal to any single matrix entry, we have $\|\cdot\|_n^\infty \leq \|\cdot\|_n^{\infty, \text{op}} \leq \|\cdot\|_n^1$. This proves the continuity of inclusion maps $\mathcal{S}_\sigma^1(X) \hookrightarrow \mathcal{S}_\ell^{\infty, \text{op}}(X) \hookrightarrow \mathcal{S}_\sigma^\infty(X)$. Since σ is constant along each matrix algebra, the \mathbf{p} -summability condition (22) is equivalent to the summability condition (6), so Theorem 2.7 tells us the three spaces are isomorphic and nuclear if the summability condition is satisfied.

Conversely, assume $\mathcal{S}_\ell^{\infty, \text{op}}(X)$ is a nuclear Fréchet space. We need to find a “Basic” Schwartz space (Definition 2.6) which is nuclear, to deduce the \mathbf{p} -summability condition (22) for ℓ . The subspace of diagonal matrices is an obvious candidate. Let Y be the countable disjoint union of diagonal sets $Y = \bigcup_{k=0}^\infty \{k\} \times \{0, \dots, p_k - 1\}$. Define scales γ on Y by $\gamma_n(k, i) = \ell_n(k)$, and embed $\theta: \mathcal{S}_\gamma^\infty(Y) \hookrightarrow \mathcal{S}_\ell^{\infty, \text{op}}(X)$, via the diagonal map $\theta(\varphi)(\alpha) = \varphi(k, i) \delta(i - j)$. For $\varphi \in \mathcal{S}_\gamma^\infty(Y)$,

$$\|\theta(\varphi)\|_n^{\infty, \text{op}} = \sup_{k \in \mathbb{N}} \ell_n(k) \|\theta(\varphi)(k)\|_{\text{op}} = \sup_{k \in \mathbb{N}} \ell_n(k) \sup_{i < p_k} |\varphi(k, i)| = \|\varphi\|_n^\infty,$$

since each $\theta(\varphi)(k)$ is a diagonal matrix. So θ is isometric in all the norms. Since a subspace of a nuclear Fréchet space is nuclear [Pietsch, 1972], Proposition 5.1.5, or [Treves, 1967], Proposition 50.1 (50.3), $\mathcal{S}_\gamma^\infty(Y)$ is nuclear.

Assume for a contradiction that every ℓ_n is not proper. Then there exists a sequence $k_1, k_2, \dots, k_j, \dots$ on which every ℓ_n is bounded on the tail. The sequence of basis elements

$e_{k_j,0}$ is then bounded by some $C_n > 0$ in each norm $\|\cdot\|_n^\infty$. By Proposition 2.1 (d), there would have to be a subsequence converging in $\mathcal{S}_\gamma^\infty(Y)$. But this is impossible.⁴

Notice that for each $k \in \mathbb{N}$ and $i < p_k$, the linear functional $\ell_n(k)e'_{ki}$ on $\mathcal{S}_\gamma^\infty(Y)$ is continuous, since $|\ell_n(k)e'_{ki}(\varphi)| = \ell_n(k)|\varphi(k,i)| \leq \|\varphi\|_n^\infty$. Also, the countable set $\{\ell_n(k)e'_{ki}\}_{k \in \mathbb{N}, i < p_k}$ converges weakly to zero in $\mathcal{S}_\gamma^\infty(Y)'$. It is an *essential* subset of the polar of the unit ball for $\|\cdot\|_n^\infty$, in the sense of [Pietsch, 1972], §2.3.1. By nuclearity and [Pietsch, 1972], 2.3.3 Theorem, for any $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ and a summing sequence c_{ki} of positive numbers so that for all $\varphi \in \mathcal{S}_\gamma^\infty(Y)$,

$$\|\varphi\|_n^\infty \leq \sum_{k \in \mathbb{N}, i < p_k} c_{ki} |\varphi(k,i)| \ell_m(k).$$

For each k_0 and i_0 , plug in $\varphi(k,i) = \delta(k-k_0, i-i_0)e_{ki}/\ell_m(k)$. The result is $\ell_n(k_0)/\ell_m(k_0) \leq c_{k_0 i_0}$. Hence we have

$$\sum_{k \in \mathbb{N}} \frac{p_k \ell_n(k)}{\ell_m(k)} \leq \sum_{k \in \mathbb{N}, i < p_k} c_{ki} < \infty.$$

Repeat the same argument to find a $p \in \mathbb{N}$ such that

$$\sum_{k \in \mathbb{N}} \frac{p_k \ell_m(k)}{\ell_p(k)} < \infty.$$

Let $C_1, C_2 > 0$ be constants bounding these respective sums. Then

$$\begin{aligned} \sum_{k \in \mathbb{N}} \frac{p_k^2 \ell_n(k)}{\ell_p(k)} &= \sum_{k \in \mathbb{N}} \frac{p_k \ell_n(k)}{\ell_m(k)} \frac{p_k \ell_m(k)}{\ell_p(k)} \\ &\leq C_1 \sup_{k \in \mathbb{N}} \frac{p_k \ell_m(k)}{\ell_p(k)} < C_1 C_2, \end{aligned} \tag{24}$$

which is the **p**-summability condition (22). □

⁴Note that $\mathcal{S}_\gamma^\infty(Y)$ is contained in $c_0(Y)$ with continuous inclusion, and clearly $\{e_{k_j,0}\}_{j \in \mathbb{N}}$ cannot have a converging subsequence in $c_0(Y)$, since they are distinct projections, and separated pairwise in sup-norm by distance 1.

Definition 6.3. If ℓ is a family of scales on \mathbb{N} , we say that ℓ is **p-regular**, where \mathbf{p} is the sequence of dimensions, if for every $n \in \mathbb{N}$ there is some $m_0 > n$ for which

$$\frac{p_k^2 \ell_n(k)}{\ell_m(k)} \geq \frac{p_{k+1}^2 \ell_n(k+1)}{\ell_m(k+1)}, \quad \text{for all } m \geq m_0, k \in \mathbb{N}. \quad (25)$$

Define two scales on \mathbb{N} by

$$\begin{aligned} \ell_{\mathbf{p}}(k) &= \max\{p_k, 1\} \\ \ell_{\text{sum}}(k) &= (1+k)p_k \end{aligned} \quad (26)$$

for $k \in \mathbb{N}$. The functions $\ell_{\mathbf{p}}$ and ℓ_{sum} may not increase with k if $p_{k+1} < p_k$ for some k . But $\ell_{\mathbf{p}}$ and ℓ_{sum} are **p-regular**, with $m_0 = n + 2$.

Proposition 6.4. *A family of scales ℓ on \mathbb{N} satisfies the **p-summability condition** (22) if $\ell_{\text{sum}} \lesssim \ell$. Conversely, if ℓ satisfies the **p-summability condition** (22) and is **p-regular**, then $\ell_{\text{sum}} \lesssim \ell$.*

Proof: If $\ell_{\text{sum}} \lesssim \ell$, then

$$\begin{aligned} \sum_{k \in \mathbb{N}} \frac{p_k^2}{\ell_m(k)} &\leq C \sum_{k \in \mathbb{N}} \frac{p_k^2}{\ell_{\text{sum}}(k)^2} \quad \text{since } \ell_{\text{sum}}(k)^2 \leq C \ell_m(k) \\ &= \sum_{k \in \mathbb{N}} \frac{p_k^2}{p_k^2 (1+k)^2} = \sum_{k \in \mathbb{N}} \frac{1}{(1+k)^2} = \frac{\pi^2}{6} < \infty, \end{aligned} \quad (27)$$

so ℓ satisfies the **p-summability condition**.

For the converse, define $\chi_{nm} = \ell_m / \mathbf{p}^2 \ell_n$ to simplify notation. Apply **p-summability** to find $m \in \mathbb{N}$ such that $\sum_{k \in \mathbb{N}} 1/\chi_{0m}(k) < \infty$. By **p-regularity**, and possibly enlarging m , we may assume $1/\chi_{0m}(k)$ decreases monotonically in k . Apply the Cauchy Condensation

Test for convergence of a series [Marsden, 1973], to see that $\sum_{i \in \mathbb{N}} 2^i / \chi_{0m}(2^i) < \infty$. Then there exists a constant $C > 1$ so that $2^i \leq C \chi_{0m}(2^i)$, $i \in \mathbb{N}$.

Next apply \mathbf{p} -summability again to find $p > m$ so that $\sum_{k \in \mathbb{N}} 1 / \chi_{mp}(k) < \infty$. Apply the Cauchy Condensation Test to see $\sum_{i \in \mathbb{N}} 2^i / \chi_{mp}(2^i) < \infty$. Let $D > 1$ be such that $2^i \leq D \chi_{mp}(2^i)$. Combining with the previous paragraph gives $2^{2i} \leq CD \chi_{0m}(2^i) \chi_{mp}(2^i)$. By \mathbf{p} -regularity, $\chi_{0m}(k)$ and $\chi_{mp}(k)$ both increase monotonically with k , so $2^{2i} \leq CD \chi_{0m}(k) \chi_{mp}(k)$ for $k \geq 2^i$. It follows that $k \leq CD \chi_{0m}(k) \chi_{mp}(k)$ for $2^i \leq k \leq 2^{2i}$. Since for $i \geq 1$, $2^{i+1} \leq 2^{2i}$, we have $k \leq CD \chi_{0m}(k) \chi_{mp}(k)$ for all $k \in \mathbb{N}$. Then $1 + k \leq (1 + CD) \chi_{0m}(k) \chi_{mp}(k) = (1 + CD) \ell_p(k) / p_k^4$ shows $\ell_{\text{sum}} \leq (1 + CD) \ell_p$ (since $p_k \leq p_k^4$) and we have $\ell_{\text{sum}} \lesssim \ell$. \square

We show that every dense nuclear two-sided Fréchet ideal is given by Theorem 6.2.

Theorem 6.5. Classification of Dense Nuclear Two-Sided Ideals. *Let a C^* -algebra B be the countable direct sum of finite dimensional C^* -algebras, with sequence of dimensions \mathbf{p} . Let A be any dense nuclear two-sided Fréchet ideal in B . Then there exists a family of scales ℓ on \mathbb{N} satisfying the \mathbf{p} -summability condition (22) such that the map $a \mapsto \{\alpha \mapsto a_\alpha\}$ gives an isomorphism of Fréchet ideals $A \cong \mathcal{S}_\ell^{\infty, \text{op}}(X)$.*

Proof: First we show that $\{e_\alpha\}_{\alpha \in X}$ is an absolute basis for A . Since $A \subset B$, every element a of A also has a unique expansion in $\{e_\alpha\}_{\alpha \in X}$. We will show the series for a converges in the Fréchet topology. For $K \in \mathbb{N}$, let $P_K = \sum_{k \leq K} 1_k$ be the sum of the units of the first K matrix algebras which make up B . By Theorem 4.6, the socle $c_f(X)$ is dense in

A. Let $\varphi \in c_f(X)$ be ϵ close to a in $\|\cdot\|_m$. For large enough K , $P_K\varphi = \varphi$, and we have

$$\begin{aligned} \left\| \sum_{k \leq K} a_\alpha e_\alpha - a \right\|_n &\leq \left\| \sum_{k \leq K} a_\alpha e_\alpha - \varphi \right\|_n + \|\varphi - a\|_n \\ &= \|P_K(a - \varphi)\|_n + \|\varphi - a\|_n \\ &\leq C_n \|a - \varphi\|_m + \|\varphi - a\|_n \leq C_n \epsilon + \epsilon, \end{aligned}$$

where we used the left ideal condition, $\|P_K\|_B = 1$, and $\|\cdot\|_n \leq \|\cdot\|_m$ in the last step.

This shows that $\{e_\alpha\}_{\alpha \in X}$ is a basis for the Fréchet algebra A . By the discussion in Definition 2.5, this basis is Schauder and equicontinuous, and by the nuclearity of A , it is also absolute.⁵

By Appendix A, we can find an equivalent family $\{\|\cdot\|_n\}_{n=0}^\infty$ of norms for the topology of A which are increasing, and satisfy $\|\cdot\|_0 = \|\cdot\|_B$ and $\|ab\|_n \leq \|a\|_n \|b\|_0$, $\|ba\|_n \leq \|b\|_0 \|a\|_n$ for all $b \in B$, $a \in A$. Use these norms to define a family of scales σ on X by $\sigma_n(\alpha) = \|e_\alpha\|_n$. Since $\|e_\alpha\|_B = 1$ for each $\alpha \in X$, $\sigma_n \geq 1$. Also $\sigma_0 = 1$ and $\sigma_n \leq \sigma_{n+1}$. By the ideal condition, $\sigma_n(\alpha) = \|e_\alpha\|_n = \|e_{k,i0} e_{k,00} e_{k,0j}\|_n \leq \|e_{k,00}\|_n = \sigma_n(k, 0, 0)$. Similarly $\sigma_n(k, 0, 0) \leq \sigma_n(\alpha)$ for any $\alpha \in X$ with first component k . Therefore the σ_n 's are constant on each matrix algebra. For each $k \in \mathbb{N}$, let $\ell_n(k)$ be the common value of $\sigma_n(k, i, j)$, $i, j < p_k$.

Apply Theorem 2.7 to see that $A \cong \mathcal{S}_\sigma^1(X) \cong \mathcal{S}_\sigma^\infty(X)$, where, by nuclearity, σ satisfies the summability condition (6), and ℓ satisfies the \mathbf{p} -summability condition (22). By Theorem 6.2, $\mathcal{S}_\ell^{\infty, \text{op}}(X)$ is isomorphic to $\mathcal{S}_\sigma^1(X) \cong \mathcal{S}_\sigma^\infty(X)$, and hence also isomorphic to A . □

⁵Note we can apply the dense ideal condition on both sides $|a_\alpha| \|e_\alpha\|_n = \|a_\alpha e_\alpha\|_n = \|e_{k,ii} a e_{k,jj}\|_n \leq C \|a\|_q$, to see directly that the basis is equicontinuous.

Remark 6.6. Automatically Involutive. Note that the ideals of Theorems 6.5 and 6.2 are involutive.

Definition 6.7. Define two scales on \mathbb{N} by

$$\begin{aligned}\ell_{\min}(k) &= \max\{p_0 + p_1 + \cdots + p_{k-1}, 1\} \\ \ell_{\max}(k) &= p_0 + p_1 + \cdots + p_k,\end{aligned}\tag{28}$$

for $k \in \mathbb{N}$. The functions ℓ_{\min} and ℓ_{\max} increase with k , but do not increase in proportion to any power of p_k .

Let S be a set and $\mathcal{S}_\sigma(S)$ be any of the Schwartz spaces in Definition 2.6 or Definition 6.1. We say that $\mathcal{S}_\sigma(S)$ is *standard Schwartz* if there exists an enumeration $\gamma: S \cong \mathbb{N}$, which induces an isomorphism of Fréchet spaces $\mathcal{S}_\sigma(S) \cong \mathcal{S}(\mathbb{N})$, where $\mathcal{S}(\mathbb{N})$ is the set of standard Schwartz functions on \mathbb{N} (see Equation (1) or Example 3.3 (b)). It is easy to check that $\mathcal{S}_\sigma(S)$ is standard Schwartz if and only if S is countably infinite and σ is equivalent to the family of scales $\{(1 + \gamma)^n\}_{n=0}^\infty$.⁶

Proposition 6.8. Let $\mathbf{p} = \{p_k\}_{k=0}^\infty$ be a sequence of dimensions. The scales $\ell_{\mathbf{p}}$, ℓ_{sum} , and ℓ_{\min} are dominated by ℓ_{\max} . The following conditions are equivalent:

- (i) $\ell_{\mathbf{p}} \lesssim \ell_{\min}$.
- (ii) $\ell_{\text{sum}} \lesssim \ell_{\min}$.
- (iii) $\ell_{\max} \lesssim \ell_{\min}$.

Let B be the C^* -algebra corresponding to \mathbf{p} . Then B contains at least one dense nuclear

⁶First use nuclearity and Theorem 2.7, or Theorem 6.2, to get $\mathcal{S}_\sigma(S) \cong \mathcal{S}_\sigma^\infty(S)$.

two-sided Fréchet ideal which is standard Schwartz, if and only if B is infinite dimensional and conditions (i)-(iii) are satisfied for some reordering of \mathbf{p} .

We say that \mathbf{p} satisfies the *growth condition* when (i) - (iii) are satisfied.

Proof: For this proof, assume that each p_k is at least 1, or equivalently that B is infinite dimensional.⁷ Clearly $\ell_{\min} \leq \ell_{\max}$ and $\ell_{\mathbf{p}} \leq \ell_{\max}$. Since $1 + k \leq \ell_{\max}(k)$, $\ell_{\text{sum}} \leq \ell_{\max}^2$, and so $\ell_{\text{sum}} \lesssim \ell_{\max}$.

Next we prove the equivalence of (i) - (iii). Since $1 \leq p_k$, we have $k \leq \ell_{\min}(k)$. Thus (i) implies $\ell_{\text{sum}} = (1 + k)p_k \leq (1 + \ell_{\min}(k))\ell_{\mathbf{p}}(k) \lesssim \ell_{\min}$. Since $\ell_{\mathbf{p}} \leq \ell_{\text{sum}}$, condition (ii) $\ell_{\text{sum}} \lesssim \ell_{\min}$ implies $\ell_{\mathbf{p}} \lesssim \ell_{\min}$, so we have (i) \Leftrightarrow (ii). The equivalence (i) \Leftrightarrow (iii) follows from $\ell_{\max} = \ell_{\min} + \ell_{\mathbf{p}}$.

Define a map $\gamma: X \rightarrow \mathbb{N}$ by $\gamma(k, i, j) = p_0^2 + \cdots p_{k-1}^2 + i + jp_k$, for $i, j < p_k$, $k \in \mathbb{N}$. Then $\gamma: X \rightarrow \mathbb{N}$ is a bijection of sets, since for a fixed k , γ increases, in steps of 1 if we move down rows, one column at a time, from $p_0^2 + \cdots p_{k-1}^2$ (at $i = j = 0$) up to $p_0^2 + \cdots p_{k-1}^2 + p_k^2 - 1$ (at $i = j = p_k - 1$). Under the bijection $\gamma: X \cong \mathbb{N}$, the scale $1 + \gamma$ on X is mapped to the scale $\sigma(q) = (1 + \gamma)(\gamma^{-1}(q)) = 1 + q$ on \mathbb{N} , resulting in a basis preserving isomorphism of $\mathcal{S}_\gamma(X)$ with standard Schwartz functions on \mathbb{N} , $\mathcal{S}_\gamma(X) \cong \mathcal{S}(\mathbb{N})$.

Now assume that the equivalent conditions (i) - (iii) hold. By (ii), $\ell_{\text{sum}} \lesssim \ell_{\min}$, so Theorem 6.2 with $\ell = \ell_{\min}$ gives us a dense nuclear two-sided Fréchet ideal $\mathcal{S}_{\min}(X) \subset B$, where $\sigma_{\min}(k, i, j) = \ell_{\min}(k)$ for $\{k, i, j\} \in X$. But $\sigma_{\min} \leq 1 + \gamma \leq \sigma_{\max}^2$, so by condition

⁷Note that on a finite set, every family of scales is equivalent to the constant scale $\sigma \equiv 1$. Also see footnote 3.

(iii), $1 + \gamma$ is a scale on X equivalent to σ_{\min} , and we have $\mathcal{S}_{\min}(X) \cong \mathcal{S}_{\gamma}(X)$, so $\mathcal{S}_{\min}(X)$ is standard Schwartz.

Conversely, assume B has a dense nuclear two-sided Fréchet ideal A which is standard Schwartz. We prove that the growth condition (i) holds. By Theorem 6.5, find a family of scales ℓ on \mathbb{N} for which $A \cong \mathcal{S}_{\ell}^{\infty, \text{op}}(X)$. Define $\sigma_n(\alpha) = \ell_n(k)$ as in Theorem 6.2. Let $\gamma : X \cong \mathbb{N}$ be an enumeration of X , such that σ is equivalent to the family of scales $\{\gamma_n\}_{n=0}^{\infty}$, where $\gamma_n = (1 + \gamma)^n$.

Define $\tilde{\ell}_n(k) = \min_{ij} \gamma_n(k, i, j)$. Reorder the sequence $\{p_k\}_{k=0}^{\infty}$ so that $\tilde{\ell}_1(0) < \tilde{\ell}_1(1) < \tilde{\ell}_1(2) < \dots$. Since γ maps X onto \mathbb{N} , $\tilde{\ell}_1(0) = 1$. The smallest the set of values $\{\gamma(0, i, j)\}_{ij < p_0}$ could be is $\{0, 1, 2, \dots, p_0^2 - 1\}$. So $\tilde{\ell}_1(1)$ can be no bigger than $p_0^2 + 1$. Similarly $\tilde{\ell}_1(k) \leq p_0^2 + p_1^2 + \dots + p_{k-1}^2 + 1$. Hence $\tilde{\ell}_1(k) \leq \ell_{\min}(k) + 1$.

Since $\sigma \sim \gamma$, there is some $n \in \mathbb{N}$ and $C_1 > 0$ such that $\gamma_1(k, i, j) \leq C_1 \ell_n(k)$, $i, j < p_k$, $k \in \mathbb{N}$. Since γ is one to one, for any $k \in \mathbb{N}$ the set of values $\{\gamma(k, i, j)\}_{ij < p_k}$ must contain a number as big as $p_k^2 - 1$. Hence $p_k^2 \leq C_1 \ell_n(k)$. So we have

$$\begin{aligned}
p_k^2 &\leq C_1 \ell_n(k) \\
&\leq C_1 C_n \tilde{\ell}_m(k) && \text{since } \ell \sim \tilde{\ell} \\
&\leq C_1 C_n (\ell_{\min}(k) + 1)^m && \text{since } \tilde{\ell}_m = \tilde{\ell}_1^m,
\end{aligned} \tag{29}$$

and (i) is satisfied. □

7 Examples

Remark 7.1. Reordering the $\{p_k\}$'s. If at least one p_k is repeated infinitely many times, then we can rearrange the sequence $\{p_k\}_{k=0}^\infty$ to satisfy the growth condition of Proposition 6.8. For example, the sequence $\{e^1, 1, e^{2^2}, 1, e^{3^3}, 1, e^{4^4}, \dots\}$ does not satisfy the growth condition because $p_0 + \dots + p_{2k-1} = k + e^1 + e^{2^2} + \dots + e^{k^k} \leq k(1 + e^{k^k})$ and $p_{2k} = e^{(k+1)^{k+1}} > e^{k^{k+1}} = (e^{k^k})^k$. Since infinitely many 1's occur in the sequence, we can however pad as many 1's as we like in between the elements $e^{(k+1)^{k+1}}$, so that after reordering $p_k < p_0 + \dots + p_{k-1}$ is satisfied.⁸

If each p_k occurs only finitely many times, the growth condition can fail, even after an optimal reordering. For example, take $p_k = e^{k^k}$. On the other hand, if the growth condition is satisfied for the sequence $\{p_k\}_{k=0}^\infty$, we can always reorder so that $p_{k-1} \leq p_k$. For let C, d be such that $p_k \leq C(p_0 + \dots + p_{k-1})^d$. If p_k is the first out of order element of the sequence, find the smallest $l > k$ for which p_l belongs at the k th spot. Redefine $p_k = p_l$, and renumber $p_{k+1} = p_k, \dots, p_l = p_{l-1}$. Since $p_l < p_k \leq C(p_0 + \dots + p_{k-1})^d$, the growth condition continues to hold. Continuing all the way up the sequence gives a nondecreasing ordering, which still satisfies the growth condition with the same constants C, d .

Example 7.2. Polynomial Growth. If $p_k \leq C(1 + k)^d$, $k \in \mathbb{N}$, then since $p_k \geq 1$, $p_0 + \dots + p_{k-1} \geq k$, and the growth condition is satisfied. The sequence $p_k = e^k$ does

⁸Note reordering the p_k 's does not change the isomorphism class of the C^* -algebra $B \cong \bigoplus_{k=0}^\infty M_{p_k}(\mathbb{C})$.

not have polynomial growth, but satisfies the growth condition since $1 + e^1 \cdots e^{k-1} = (e^k - 1)/(e - 1)$, so $p_k \leq (e - 1)(1 + p_0 + \cdots p_{k-1})$.

Remark 7.3. Constructing Fréchet Ideals. A generalization of a construction used in the proof of Proposition 6.8 seems worthy of note. Let ℓ and σ be as in Theorem 6.2, with the \mathbf{p} -summability condition (22) satisfied. Pick a family of scales β on X so that $\ell_n(k) \leq \beta_n(i, j, k)$. Also insure that for each $n \in \mathbb{N}$ there is a sufficiently large $m \in \mathbb{N}$ so that $\beta_n(i, j, k) \leq \ell_m(k)$. Then clearly $\sigma \sim \beta$. So $\mathcal{S}_\beta(X)$ is a dense nuclear two-sided Fréchet ideal in the C^* -algebra B , by its isomorphism with $\mathcal{S}_\sigma(X)$.

Example 7.4. $C^\infty(G)$, G Compact Connected Lie Group. By [Sug, 1971], Theorem 4, the Fréchet space $C^\infty(G)$ is isomorphic to standard Schwartz functions on D , where D is the set of all dominant G -integral forms on the Lie algebra of a maximal toral subgroup of G . For example, consider the circle group $G = \mathbb{T}$. The Fréchet space $C^\infty(\mathbb{T})$ is topologized by seminorms $\|\varphi\|_i = \|\partial^i \varphi\|_\infty$, $\varphi \in C^\infty(\mathbb{T})$, where $\partial = \frac{d}{d\theta}$. The Fourier transform $\widehat{\cdot}$ changes $C^\infty(\mathbb{T})$ into $\mathcal{S}(\mathbb{Z})$, $C^*(\mathbb{T})$ into $c_0(\mathbb{Z})$, and convolution multiplication of functions into pointwise multiplication. The transformed seminorms are $\|\widehat{\varphi}\|_i = \sup_{k \in \mathbb{Z}} |k^i \widehat{\varphi}(k)|$.

Example 7.5. C^∞ functions on the Cantor group. The Cantor group K is a compact locally compact group, which is not a Lie group. It is totally disconnected and can be described as the dual group of the discrete abelian group $\widehat{K} = \mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$ of dyadic rationals from 0 to 1. Define a proper scale $\sigma(\frac{l}{2^p}) = 2^p$ on \widehat{K} , for any $p \in \mathbb{N}$. Here l

is any positive odd number less than 2^p . The additive identity is 0, when $p = 0$ and $l = 0$ or $l = 1$. Let $A = \mathcal{S}_\sigma^\infty(\widehat{K})$ be σ -rapidly vanishing functions on \widehat{K} , with sup-norm. A is a dense Fréchet ideal in $B = c_0(\widehat{K})$, the C^* -algebra of functions vanishing at ∞ on \widehat{K} , with pointwise multiplication. (Note that convolution multiplication on K becomes pointwise multiplication on \widehat{K} via the Fourier transform.) Since σ satisfies the summability condition, A is also nuclear [Sch, 1998], Lemma 1.2 with $q = 2$, Theorem 1.6.

We show that A is isomorphic to standard Schwartz functions $\mathcal{S}(\mathbb{N})$. Let $\gamma: \widehat{K} \rightarrow \mathbb{N}$ be the map $\gamma(\frac{l}{2^p}) = 2^{p-1} + \lfloor l/2 \rfloor$, where $\lfloor \cdot \rfloor$ is the largest integer not greater than, and we set $\gamma(0) = 0$. For each $p \in \mathbb{N}^+$, note that $\lfloor l/2 \rfloor$ goes from 0 to $2^{p-1} - 1$, in increments of 1, and we have $\gamma < \sigma$. Also $\sigma \leq 2\gamma$ except at $p = 0$. So $1 + \gamma$ and σ are equivalent scales on \widehat{K} . Since γ is a bijection $\gamma: \widehat{K} \cong \mathbb{N}$, an isomorphism $A \cong \mathcal{S}(\mathbb{N})$ is determined by mapping δ_α to $\delta_{\gamma(\alpha)}$ for each $\alpha \in \widehat{K}$.

A Appendix. Refining the Ideal Condition, and m -Convexity

We show that the constants C_n can always be taken equal to 1 in the dense ideal inequality (12), and also $m_n = n$, by passing to an equivalent family of seminorms. We also note

that any Fréchet algebra satisfying the ideal inequality is m -convex.⁹

Let A be a right Fréchet ideal in a Banach algebra B , and let $\{\|\cdot\|_n\}_{n=0}^\infty$ be an increasing family of norms giving the topology for A , with $\|\cdot\|_0 = \|\cdot\|_B$, as in Definition 3.1. Define a new family of seminorms by

$$\|a\|_n^* = \sup\{\|ab\|_n \mid \|b\|_0 \leq 1, b \in B\}, \quad (30)$$

for $a \in A$. Using the right ideal inequality (12) we see that $\|a\|_n^* \leq C_n \|a\|_m$, so the topology given by $\{\|\cdot\|_n^*\}_{n=0}^\infty$ is dominated by the original topology on A . These new seminorms satisfy the right ideal inequality with $m_n = n$ and $C_n = 1$:

$$\begin{aligned} \|ab\|_n^* &= \sup\{\|abb_1\|_n \mid \|b_1\|_0 \leq 1, b_1 \in B\} && \text{by definition (30)} \\ &\leq \sup\{\|ab_2\|_n \mid \|b_2\|_0 \leq \|b\|_0, b_2 \in B\} && b_2 = bb_1, \text{ so } \|b_2\|_0 \leq \|b\|_0 \\ &= \|b\|_0 \sup\{\|ab_3\|_n \mid \|b_3\|_0 \leq 1, b_3 \in B\} && b_3 = b_2/\|b\|_0 \\ &= \|a\|_n^* \|b\|_0 && \text{by definition (30)}. \end{aligned} \quad (31)$$

Note we used the submultiplicativity of the norm $\|\cdot\|_0$ on B in the second step. The inequality

$$\|ab\|_n \leq \|a\|_n^* \|b\|_0 \quad (32)$$

can also be verified from the definition (30).

Since B may not be unital, the seminorms $\{\|\cdot\|_n^*\}_{n=0}^\infty$ are not necessarily equivalent to our original family. (For example, B could be a radical Banach algebra with $ab = 0$ for all $a, b \in B$, in which case $\|a\|_n^* = 0$ for all n .) So define new norms by $\|a\|_n^{*+} =$

⁹A Fréchet algebra A is m -convex if it can be topologized by a family of *submultiplicative* seminorms, i.e. ones that satisfy $\|a_1 a_2\|_n \leq \|a_1\|_n \|a_2\|_n$ for $a_1, a_2 \in A$.

$\max\{\|a\|_n^*, \|a\|_n\}$, $a \in A$. Note that $\|\cdot\|_0^{*+} = \|\cdot\|_0 = \|\cdot\|_B$, and $\|ab\|_n^{*+} \leq \|a\|_n^* \|b\|_0$ by our estimates above. The new family $\{\|\cdot\|_n^{*+}\}_{n=0}^\infty$ gives a topology equivalent to the original family $\{\|\cdot\|_n\}_{n=0}^\infty$, and satisfies the right ideal inequality $\|ab\|_n^{*+} \leq \|a\|_n^{*+} \|b\|_0$. Hence for any Banach algebra B with right Fréchet ideal A , it is always possible to find an equivalent family of norms giving the topology on A , such that the new norms satisfy the right Fréchet ideal inequality (12) with $C_n = 1$ and $m_n = n$ for every $n \in \mathbb{N}$, and the zeroth norm $\|\cdot\|_0$, which is the norm $\|\cdot\|_B$ on B , remains unchanged.

The new norms are submultiplicative for every n : $\|a_1 a_2\|_n^{*+} \leq \|a_1\|_n^{*+} \|a_2\|_0^{*+} \leq \|a_1\|_n^{*+} \|a_2\|_n^{*+}$, $a_1, a_2 \in A$. Hence dense Fréchet ideals are always m -convex Fréchet algebras.

The same results hold for left ideals, by switching the order in the above arguments. In the left case, we define $\|a\|_n^\dagger = \sup\{\|ba\|_n \mid \|b\|_0 \leq 1, b \in B\}$, and $\|a\|_n^{\dagger+} = \max\{\|a\|_n^\dagger, \|a\|_n\}$.

Finally, assume that A is a two-sided Fréchet ideal in the Banach algebra B . As before, A is topologized by increasing norms $\{\|\cdot\|_n\}_{n=0}^\infty$, with $\|\cdot\|_0 = \|\cdot\|_B$, and left and right ideal inequalities are satisfied: $\|ab\|_n \leq C_n \|a\|_m \|b\|_0$, $\|ab\|_m \leq C_m \|a\|_k \|b\|_0$, and $\|ba\|_n \leq C_n \|b\|_0 \|a\|_m$, $\|ba\|_m \leq C_m \|b\|_0 \|a\|_k$, for all $a \in A$ and $b \in B$. Define a new family of seminorms by

$$\|a\|_n^{two} = \sup\{\|cab\|_n \mid \|c\|_0, \|b\|_0 \leq 1, c, b \in B\}, \quad (33)$$

for $a \in A$. Using the right and left ideal inequalities we see that $\|a\|_n^{two} \leq C_n C_m \|a\|_p$, so the topology given by $\{\|\cdot\|_n^{two}\}_{n=0}^\infty$ is dominated by the original topology on A . These

new seminorms satisfy the right ideal inequalities with $m_n = n$ and $C_n = 1$:

$$\begin{aligned}
\|ab\|_n^{two} &= \sup\{ \|cabb_1\|_n \mid \|c\|_0, \|b_1\|_0 \leq 1, c, b_1 \in B \} && \text{by definition (33)} \\
&\leq \sup\{ \|cab_2\|_n \mid \|c\|_0 \leq 1, \|b_2\|_0 \leq \|b\|_0, c, b_2 \in B \} && b_2 = bb_1, \text{ so } \|b_2\|_0 \leq \|b\|_0 \\
&= \|b\|_0 \sup\{ \|cab_3\|_n \mid \|c\|_0, \|b_3\|_0 \leq 1, c, b_3 \in B \} && b_3 = b_2/\|b\|_0 \\
&= \|a\|_n^{two} \|b\|_0 && \text{by definition (33)}. \quad (34)
\end{aligned}$$

Again we used the submultiplicativity of the norm $\|\cdot\|_0$ on B in the second step. Similarly, the left ideal inequality holds: $\|ba\|_n^{two} \leq \|b\|_0 \|a\|_n^{two}$, $a \in A$, $b \in B$.

Since the seminorms $\{\|\cdot\|_n^{two}\}_{n=0}^\infty$ could give a weaker topology on A (as we discussed in the right ideal case above), we define our final set of norms on A by $\|a\|_n^{two+} = \max\{\|a\|_n^{two+}, \|a\|_n^*, \|a\|_n^\dagger, \|a\|_n\}$. Note that $\|\cdot\|_0^{two+} = \|\cdot\|_0 = \|\cdot\|_B$. We have the four inequalities:

$$\begin{aligned}
\|ab\|_n^{two} &\leq \|a\|_n^{two} \|b\|_0 && \text{by (34)} \\
\|ab\|_n^* &\leq \|a\|_n^* \|b\|_0 && \text{by (31)} \\
\|ab\|_n^\dagger &= \sup\{ \|cab\|_n \mid \|c\|_0 \leq 1, c \in B \} && \text{by definition of } \|\cdot\|_n^\dagger \text{ above} \\
&\leq \|a\|_n^{two} \|b\|_0 && \text{by definition (33) of } \|\cdot\|_n^{two} \\
\|ab\|_n &\leq \|a\|_n^* \|b\|_0, && \text{by (32)}
\end{aligned}$$

for $a \in A$ and $b \in B$. Putting these together shows that the norms $\{\|\cdot\|_n^{two+}\}_{n=0}^\infty$ satisfy the right ideal inequality with $C_n = 1$, $m_n = n$ for every $n \in \mathbb{N}$. Similarly, they satisfy the left ideal inequalities with the same constraints.

We have shown that for any Banach algebra B with two-sided Fréchet ideal A , it is always possible to find an equivalent family of norms giving the topology on A , such

that the new norms satisfy the right and left Fréchet ideal inequalities with $C_n = 1$ and $m_n = n$ for every $n \in \mathbb{N}$, and the zeroth norm $\|\cdot\|_0$, which is the norm $\|\cdot\|_B$ on B , remains unchanged.

B Appendix. Counterexamples

Example B.1. Standard Schwartz Functions Not an Ideal. When the sequence of dimensions $\mathbf{p} = \{p_k\}_{k=0}^\infty$ doesn't satisfy the growth condition, Proposition 6.8 tells us a dense nuclear Fréchet ideal $\mathcal{S}_\sigma(X)$ given by Theorem 6.2 is never standard Schwartz.

Assume the growth condition is violated, and cannot be repaired by reordering \mathbf{p} . Then \mathbf{p} cannot have a bounded subsequence by Remark 7.1 and so must be proper, and we can arrange $p_k \leq p_{k+1}$, $k \in \mathbb{N}$. Let $\gamma(k, i, j) = p_0^2 + \cdots p_{k-1}^2 + i + jp_k$, $\{k, i, j\} \in X$ be as in the proof of Proposition 6.8. We saw that $\mathcal{S}_\gamma(X)$ is standard Schwartz. We show directly that $\mathcal{S}_\gamma(X)$ is not an ideal in B . Define a scale β on X by $\beta(k, i, j) = (1 + k)p_{k-1}(i + 1)(jp_k + 1)$. Then $\beta \sim 1 + \gamma$, and our calculations will simplify using β in place of γ . For each $k \in \mathbb{N}$, let $c_1(k)$ be 1's in the first column:

$$c_1(k) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in M_{p_k}(\mathbb{C}).$$

Then $\|c_1(k)\|_{\text{op}} = \sqrt{p_k}$, so for each $K \in \mathbb{N}$,

$$S_K = \sum_{k=0}^K \frac{c_1(k)}{\sqrt{p_k}} \in c_f(X)$$

satisfies $\|S_K\|_B = 1$. The n th norm in $\mathcal{S}_\beta^1(X)$ is

$$\begin{aligned} \|S_K\|_n &= \sum_{k=0}^K \sum_{i=0}^{p_k-1} \frac{\beta(k, i, 0)^n}{\sqrt{p_k}} \\ &= \sum_{k=0}^K \frac{(1+k)^n p_{k-1}^n}{\sqrt{p_k}} \sum_{i=0}^{p_k-1} (1+i)^n \\ &\geq \sum_{k=0}^K \frac{(1+k)^n p_{k-1}^n}{\sqrt{p_k}} p_k^n \geq p_K^{n-1/2}. \end{aligned}$$

And for each $K \in \mathbb{N}$,

$$T_K = \sum_{k=0}^K e_{k,00} \in c_f(X)$$

is an element of $\mathcal{S}_\beta(X)$ with m th norm equal to

$$\begin{aligned} \|T_K\|_m &= \sum_{k=0}^K \beta(k, 0, 0)^m \\ &= \sum_{k=0}^K (1+k)^m p_{k-1}^m \leq K(1+K)^m p_{K-1}^m. \end{aligned}$$

Since $c_1(k)e_{k,00} = c_1(k)$, $S_K T_K = S_K$. To see that the left ideal condition is violated,

note that

$$\begin{aligned} \frac{\|S_K T_K\|_n}{\|S_K\|_B \|T_K\|_m} &= \frac{\|S_K\|_n}{\|T_K\|_m} \\ &\geq \frac{p_K^{n-1/2}}{K(1+K)^m p_{K-1}^m} \\ &\geq \frac{p_K^{n-1/2}}{2^m (p_0 + \cdots p_{K-1})^{1+m} p_{K-1}^m}, \end{aligned} \tag{35}$$

since $K \leq p_0 + \cdots p_{K-1}$ and $1+K \leq 2K$ for $K \geq 1$. If for some $n \geq 1$, there were an $m > n$ for which the right hand side of (35) were bounded for all $K \in \mathbb{N}$, then the growth condition on the sequence \mathbf{p} would have to hold.

Example B.2. A Banach Algebra Not an Ideal in $c_0(\mathbb{N})$. When the standard basis $\{\delta_k\}_{k=0}^\infty$ is an absolute basis for a dense Fréchet subspace F of the commutative C^* -algebra $c_0(\mathbb{N})$, then F is isomorphic to $\mathcal{S}_\sigma^1(\mathbb{N})$ (Theorem 2.7), and is easily seen to be an ideal in $c_0(\mathbb{N})$ by the same argument as Example 3.3 (a). We exhibit a dense Banach subalgebra A of $c_0(\mathbb{N})$, for which $\{\delta_k\}_{k=0}^\infty$ is not an absolute basis, and such that A is not an ideal in $c_0(\mathbb{N})$.

Let σ be a proper scale on \mathbb{N} . Define a norm by

$$\|f\|_{\sigma,1} = \sup_{k \in \mathbb{N}} \left(\sigma(k) \max \{ |f_+(k)|, \sigma(k)|f_-(k)| \} \right), \quad (36)$$

where $f_+(k) = f(2k) + f(2k+1)$, $f_-(k) = f(2k) - f(2k+1)$, and $f \in c_f(\mathbb{N})$. Define a new scale β , which is “half of σ ” by $\beta(2k) = \beta(2k+1) = \sigma(k)$. Define two related norms $\|f\|_\beta = \|\beta f\|_\infty$ and $\|f\|_{\beta^2} = \|\beta^2 f\|_\infty$, which topologize the Banach algebras $c_0(\mathbb{N}, \beta)$ and $c_0(\mathbb{N}, \beta^2)$, respectively. Let A be the completion of $c_f(\mathbb{N})$ in the norm $\|\cdot\|_{\sigma,1}$. Then $c_f(\mathbb{N}) \subseteq c_0(\mathbb{N}, \beta^2) \hookrightarrow A \hookrightarrow c_0(\mathbb{N}, \beta) \hookrightarrow c_0(\mathbb{N})$, where the inclusions \hookrightarrow are continuous.

Note that $\|\delta_{2k}\|_{\sigma,1} = \|\delta_{2k}\|_{\beta^2} = \sigma(k)^2$ and $\|\delta_{2k+1}\|_{\sigma,1} = \|\delta_{2k+1}\|_{\beta^2} = \sigma(k)^2$. But $\|\delta_{+,k}\|_{\sigma,1} = \|\delta_{+,k}\|_\beta = 2\sigma(k)$ and $\|\delta_{-,k}\|_{\sigma,1} = \|\delta_{-,k}\|_{\beta^2} = 2\sigma(k)^2$, where $\delta_{+,k}$ and $\delta_{-,k}$ are defined analogously to f_+ and f_- . Then

$$\frac{\|\delta_{+,k} * \delta_{-,k}\|_{\sigma,1}}{\|\delta_{+,k}\|_{\sigma,1} \|\delta_{-,k}\|_\infty} = \frac{\|\delta_{-,k}\|_{\sigma,1}}{\|\delta_{+,k}\|_{\sigma,1} \|\delta_{-,k}\|_\infty} = \frac{2\sigma(k)^2}{2\sigma(k) * 1} = \sigma(k), \quad (37)$$

which tends to ∞ as $k \rightarrow \infty$, since σ was assumed proper. So A is not an ideal in $c_0(\mathbb{N})$.

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